Algebra Review Information

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Types of Numbers

- **Natural Numbers**: The set of natural numbers is denoted by $\mathbb{N}$. It is the set of "counting numbers:"
$$1, 2, 3, 4, 5, 6, 7, ...$$
Sometimes 0 is included.

- **Integers**: The set of integers is denoted by $\mathbb{Z}$. It is the set of "whole numbers:"
$$..., -3, -2, -1, 0, 1, 2, 3, 4, ...$$
Every natural number is also an integer.

- **Rational Numbers**: The set of rational numbers is denoted by $\mathbb{Q}$. A number is rational if it *can* be written as a ratio (fraction) of integers.
Every integer n is also rational (it can be written as $\frac{n}{1}$).
Examples: $\frac{1}{7}$, $\frac{491}{2}$, 0.3 (this is $\frac{3}{10}$), -0.111111111... (this is $-\frac{1}{9}$), 8 (this is $\frac{8}{1}$), $\sqrt[4]{81}$ (this is $\frac{3}{1}$), $\sqrt[3]{-64}$ (this is $\frac{-4}{1}$)

- **Real Numbers**: The set of real numbers is denoted by $\mathbb{R}$. Any number you can think of is real, as long as it doesn't involve the even root of a negative number. Every real number is either rational or irrational, so for examples of real numbers, see the rational and irrational numbers.

- **Irrational Numbers**: Any real number that *cannot* be expressed as a ratio of integers is called irrational.
Examples: $\sqrt{2}$, π, $\frac{2\sqrt{3}}{7}$, e, $\sqrt[3]{-31}$

- **Imaginary Numbers**: An imaginary number is any number that *can* be written in the form bi where b is a real number and $i = \sqrt{-1}$.
Examples: $4i$, $\sqrt{-87}$, $\frac{\sqrt{3}i}{2}$, $\frac{3i}{2}$, $\sqrt[82]{-4}$

- **Complex Numbers**: The set of complex numbers is denoted by $\mathbb{C}$. A complex number is any number that *can* be written in the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$.
Every real number is also complex (if a is real, then $a = a + 0i$).
Every imaginary number is also complex (if bi is imaginary, then $bi = 0 + bi$).

The Domain of a Variable or Function

Recall that the domain of a variable is the set of all values for which both sides of the equation is *defined*.

It is also important to remember that a value that is in the domain of a variable does **not** have to make the equation *true*. For example, 3 is in the domain of the variable x for the equation $x - 7 = 1$, but if you plug in 3 for x, the equation does not create a true statement. (A value in the domain of a variable that makes the equation *true* is called a solution.)

For the purposes of this class, unless we state otherwise, the domain of a variable will be the set of all **REAL** values for which both sides of the equation is **REAL**. So, we will start by "hoping" the domain of a variable is "all real numbers" (or $\mathbb{R}$ or $(-\infty, \infty)$) and then throwing out the values that "don't work."

The domain of a function $f(x)$ will be the domain of the variable x.

The major issues to consider when identifying the domain of a variable are (1) radicals and (2) division/fractions (and later, logarithms). We will use the following rules to find the domain of a variable in an equation:

1. In the real number system, you cannot take the even root of a negative number. So, the domain of the expression $\sqrt[a]{u}$ if a is even is the solution to the equation $u \geq 0$. (This means u cannot be negative.)

2. Any expression divided by 0 is UNDEFINED (it's worse than "not real:" it does not exist at all!). So, the domain of the expression $\frac{u}{v}$ is the solution to the equation $v \neq 0$.

Interval Notation

Things to remember about using interval notation:

- To join two intervals together, and say that our solution or domain can be in either one of the intervals, we use the $\cup$ symbol. It means "union" and you can think of it as "or."
  Example: $(-3, 0] \cup [1, \infty)$ means our value can either be between -3 and 0 or it can be anything bigger than or equal to 1.

- An open (round) parenthesis is like an open circle on a graph - the value is **NOT** included.
  Example: $(-3, 0)$ means any number between -3 and 0 but NOT including -3 or 0.

- A closed (square) bracket is like a closed circle on a graph - the value **IS** included.
  Example: $[-3, 0]$ means any number between -3 and 0, including -3 and 0.

- We ALWAYS use open parentheses with ∞ or $-\infty$.
  Example: We write $(-\infty, 3) \cup (3, \infty)$. We NEVER write $[-\infty, 3) \cup (3, \infty]$.

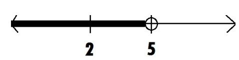
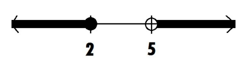
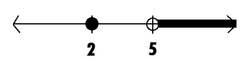
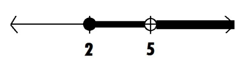
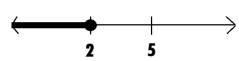
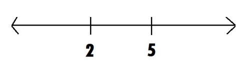
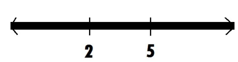
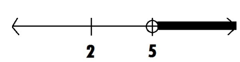
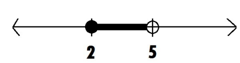
- To indicate that your value must be in BOTH segments/intervals, we use the $\cap$ symbol. It means "intersection" and you can think of it as "and."
  Example: $(-3, 0] \cap (-1, \infty) = (-1, 0]$ because only numbers between -1 and 0 are in BOTH of the intervals $(-3, 0]$ AND $(-1, \infty)$.

Exercises

Find the domain of the variable x in the following equations:

1. $3x + 4 = 22$
2. $\dfrac{3x+4}{x-1} = 22$
3. $\sqrt{3x+4} = \dfrac{22}{x-1}$

Now let's talk about translating logical statements into inequalities. Match the statements below with the intervals and graphs:

(1) $x > 5$ and $x \leq 2$	(I) $(-\infty, 5)$	(A) 
(2) $x < 5$ and $x \leq 2$	(II) $(-\infty, 2]$	(B) 
(3) $x > 5$ or $x \leq 2$	(III) $(5, \infty)$	(C) 
(4) $x < 5$ or $x \leq 2$	(IV) $\emptyset$	(D) 
(5) $x > 5$ and $x \geq 2$	(V) $(-\infty, \infty)$	(E) 
(6) $x > 5$ or $x \geq 2$	(VI) $[2, 5)$	(F) 
(7) $x < 5$ and $x \geq 2$	(VII) $(-\infty, 2] \cup (5, \infty)$	(G) 
(8) $x < 5$ or $x \geq 2$	(VIII) $[2, \infty)$	(H) 
		(I) 
		(J) 

Dealing with Fractions

Let a, b, c, d be any algebraic expressions (meaning they could just be numbers, or they could be expressions containing a variable).

- Adding and subtracting fractions: (1) Get a common denominator. (2) Add/subtract the numerators.
  - $\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{cb}{bc} = \frac{ad \pm cb}{bd}$
  - Example with numbers: $\frac{1}{8} - \frac{12}{7} = \frac{7}{56} - \frac{96}{56} = -\frac{89}{56}$
  - Example with variables:

$$\begin{aligned}\frac{3}{(x-7)(1-x)} + \frac{x-1}{(x-7)(x+4)} &= \frac{3(x+4)}{(x-7)(1-x)(x+4)} + \frac{(x-1)(1-x)}{(x-7)(1-x)(x+4)} \\ &= \frac{11+x-x^2}{(x-7)(1-x)(x+4)}\end{aligned}$$

- Multiplying fractions: No common denominator necessary. Multiply the numerator and multiply the denominator.
  - $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
  - Example with numbers: $\frac{3}{5} \cdot \frac{8}{11} = \frac{24}{55}$
  - Example with variables: $\frac{x-12}{x+1} \cdot \frac{x-1}{x-3} = \frac{(x-12)(x-1)}{(x+1)(x-3)}$
- Dividing fractions: No common denominator necessary. Invert and multiply.
  - $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ or $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$
  - Example with numbers: $\frac{3}{5} \div \frac{8}{11} = \frac{3}{5} \cdot \frac{11}{8} = \frac{33}{40}$
  - Example with variables: $\frac{x-12}{x+1} \div \frac{x-1}{x-3} \frac{x-12}{x+1} \cdot \frac{x-3}{x-1} = \frac{(x-12)(x-3)}{(x+1)(x-1)}$
- Cancellation and fractions: You can only cancel FACTORS, never TERMS. This means you can only cancel when you are MULTIPLYING inside the fraction not when you are ADDING inside the fraction.

- Yes: $\dfrac{3(x-1)}{(2x+1)(x-1)} = \dfrac{3}{2x+1}$
  Here, the top and bottom are both written as FACTORS (multiplication only) so the common factor of $(x-1)$ can be "cancelled"
- NO!!!!!! $\dfrac{3x+17}{x^2+17} = \dfrac{3+17}{x+17} = \dfrac{3}{x}$
  Here, you are ADDING two things in the numerator and you are ADDING two things in the denominator. So, you cannot cancel the x and you cannot cancel the 17.
- Pro tip: Think about it with numbers. $\dfrac{12}{20} = \dfrac{4\cdot 3}{4\cdot 5} = \dfrac{3}{5}$ (you can cancel a FACTOR of 4) but $\dfrac{12}{20} = \dfrac{13-1}{13+7} \neq \dfrac{-1}{7}$ - you can't cancel when you write it as a sum.

Polynomials: Dividing, Factoring, and Finding Roots

Definition. A function p is a polynomial function if it is of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_2x^2 + a_1x + a_0$$

for some positive integer n and real numbers $a_0, a_1, ..., a_n$, with $a_n \neq 0$. The a_i are called the coefficients of p and n is called the degree of p.
Informally, p is a polynomial if it is just (constant multiples of) powers of x added together, and all the powers of x that appear are whole numbers. The highest power of x that appears is called the degree and the numbers in front of the powers of x are called the coefficients.

- Add polynomials by adding like terms: $a_ix^i + b_ix^i = (a_i + b_i)x^i$ (only add if the power of x is the same)
- Subtract polynomials by distributing the minus sign and then add
- Multiply polynomials by using the distributive law (general form of FOIL)
- Dividing polynomials: Let p and q be polynomial functions. To find $p \div q$, use polynomial long division. Synthetic division is also an option, but only if $q(x) = x - a$. If you are dividing by a polynomial with degree $n > 1$ or with a lead coefficient $a_n \neq 1$, then you CANNOT use synthetic division.
  - **Definition.** If q divides evenly into p, meaning that the remainder when you compute $p \div q$ is zero, then we call q a factor of p.

Definition. A number a is a root of a polynomial function p if $p(a) = 0$. If a is a *real* root of p then $(a, 0)$ is also an x-intercept of the graph of p.

Theorem 1: Let p be a polynomial function. A number a is a root of p if and only if $(x - a)$ is a factor of p.

So, root-finding problems are the same as factoring problems! (Bonus challenge: prove this theorem using only the definitions in this section.)

Generally speaking, when we say in an algebraic context "factor the polynomial p," we mean write p as a product of factors with the lowest possible degrees such that all coefficients of all factors are integers.

We can also "factor over $\mathbb{Q}$" or "factor over $\mathbb{R}$" or "factor over $\mathbb{C}$," where you factor p into factors with the lowest degree possible when you allow coefficients to come from the rational numbers, real numbers, or complex numbers, respectively.

As an example, consider $p(x) = 3x^6 + 6x^5 - 3x^4 + 22x^3 + 48x^2 - 8x - 16$.

$$\begin{aligned}
p(x) &= 3x^6 + 6x^5 - 3x^4 + 22x^3 + 48x^2 - 8x - 16 && (1)\\
&= (3x^3 + 6x^2 - x - 2)(x^3 + 8) && (2)\\
&= (3x^2 - 1)(x + 2)(x^3 + 8) && (3)\\
&= (3x^2 - 1)(x + 2)(x + 2)(x^2 + 2x + 4) && (4)\\
&= 3\left(x^2 - \frac{1}{3}\right)(x + 2)^2(x^2 + 2x + 4) && (5)\\
&= 3\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right)(x + 2)^2(x^2 + 2x + 4) && (6)\\
&= 3\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right)(x + 2)^2(x - (-1 + i\sqrt{3}))(x - (-1 - i\sqrt{3})) && (7)\\
& && (8)
\end{aligned}$$

Notes:

1. This line is the definition of $p(x)$.
2. Here p is written as a product of two cubic polynomials, but both factor into products of polynomials with smaller degrees with integer coefficients so we are definitely not done.
3. We can get from the previous line to this line using factoring by grouping.

4. We can get from the previous line to this line using the sum of cubes factoring formula.

    - This is the factorization of p over $\mathbb{Z}$ (the integers). This would be our final answer if the instructions were to "factor p" with no further specifications. This is also an appropriate answer to "factor p over $\mathbb{Q}$."

5. On this line, we've factored a 3 out of the first factor.

    - This is an alternative answer to "factor p over $\mathbb{Q}$." The degrees of the factors haven't changed, so it is equivalent to the answer on the previous line, and our coefficients are still in $\mathbb{Q}$, although they are no longer all in $\mathbb{Z}$.

6. Here it is possible to factor $x^2 - \frac{1}{3}$ as a difference of squares and keep our coefficients in $\mathbb{R}$.

    - This is an appropriate answer to "factor p over $\mathbb{R}$." All of our coefficients are in $\mathbb{R}$, and every factor is degree 1 (lowest possible degree) except for the last one, which does not factor over $\mathbb{R}$.

7. Finally, if we extend to $\mathbb{C}$, we can factor the last quadratic factor as a product of two linear factors. This is as far as we can go and we have now "factored p over $\mathbb{C}$."

In general, how far can you go? When are you done factoring? Here are several facts/ theorems:

- Every odd degree polynomial has at least one real root.
- Every polynomial with real coefficients factors over $\mathbb{R}$ as a product of linear factors and irreducible quadratic factors.
    - A quadratic (degree 2) polynomial is <u>irreducible</u> if its discriminant $D = b^2 - 4ac$ is negative.
- If a polynomial with integer coefficients has a rational root, then it factors over $\mathbb{Z}$. If a polynomial with rational coefficients has a rational root, then it factors over $\mathbb{Q}$.
    - **The Rational Roots Theorem:** The only possible rational roots of a polynomial with integer coefficients with constant term a_0 and lead coefficient a_n are of the form $\frac{p}{q}$ where p divides a_0 and q divides a_n (and divisors can be positive or negative).

  - A quadratic (degree 2) polynomial factors over $\mathbb{Z}$ if its discriminant $D = b^2 - 4ac$ is a perfect square (a square of an integer - examples: $D = 0, 1, 4, 9, 16, 25, 36, 49, ...$).

- **The Conjugate Pairs Theorem:** If a polynomial has real coefficients and $a + bi$ is a root, then $a - bi$ is a root.

- **The Fundamental Theorem of Algebra:** Every polynomial of degree n (no matter where the coefficients come from) factors into a product of n linear factors over $\mathbb{C}$. (Alternatively: Every polynomial of degree n has n (not necessarily distinct) roots.)

Definition. If $p(x)$ is a polynomial function with $p(x) = (x - a)^m q(x)$ and $q(a) \neq 0$, then we call m the multiplicity of the root a. (The multiplicity of a root a of a polynomial p is the highest power of $(x - a)$ you can factor out of p.)

In the previous example, $a = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -1 + i\sqrt{3}, -1 - i\sqrt{3}$ are all roots of p with multiplicity 1 and $a = -2$ is a root of p with multiplicity 2. All together, there are 6 roots of p (-2 is counted twice because its multiplicity is 2) and the degree of p is 6. This shows an application of the Fundamental Theorem of Algebra. If we had stopped factoring at any step before our last one, the Fundamental Theorem of Algebra would have told us that we had not completely factored p over $\mathbb{C}$ yet.

Exercises

For each polynomial below:
(a) State the degree and lead coefficient.
(b) Find all (i) rational, (ii) real, and (iii) complex roots of the polynomial.
(c) State the multiplicity of each root.
(d) Factor over $\mathbb{Z}$.
(e) Factor over $\mathbb{R}$.
(f) Factor over $\mathbb{C}$.
(g) Find the x and y intercepts.
(h) Find the end behavior: $\lim_{x\to-\infty} p(x)$ and $\lim_{x\to\infty} p(x)$.

1. $p(x) = (x+1)^2(x-1)^3$
2. $p(x) = -5\left(x+\frac{2}{3}\right)\left(x-\frac{1}{2}\right)^2$
3. $p(x) = -x^4 + 6x^3 - 9x^2$
4. $p(x) = -2x^3 + 4x^2 - 4x + 8$
5. $p(x) = x^4 - 8x^3 + 18x^2 + 54x + 27$
6. $p(x) = x^3 - x^2 - 4x + 4$
7. $p(x) = x^3 + x^2 + 2x + 2$
8. $p(x) = x^4 + 6x^3 + 14x^2 + 16x + 8$ Hint: $-1+i$ is one of the zeros.
9. $p(x) = 3x^4 + 5x^3 + x^2 + 5x - 2$
10. $p(x) = 3x^3 - x^2 - 6x + 3$
11. $p(x) = (x-1)(x-2i)(x+2i)(2x-6i)(2x+6i)$
12. $p(x) = x^5 - 5x^4 + 2x^3 + 22x^2 - 20x$ Hint: $3-i$ is one of the zeros.
13. $p(x) = x^3 - 5x^2 + 7x + 13$
14. $p(x) = 2x^4 - 10x^3 + 23x^2 - 24x + 9$
15. $p(x) = x^5 - 2x^4 - x^3 + 8x^2 - 10x + 4$

Find the remaining roots of the polynomial described (if any) and then find an equation of a polynomial with real coefficients that satisfies the description.

1. Degree 3, Zeros 2 and $3 + i$.
2. Degree 6, Zeros $2i, 4 + i, i - 1$.
3. Degree 4, Zeros $2 + 3i, 1 - 4i$, Lead Coeff -3.
4. Degree 6, Zeros 3, 0 (with mult 3), $2 - 3i$, Lead Coeff 4.
5. Degree 4, Zeros 1, -1, $3 + i$, y-intercept 20.

Properties and Laws of Exponents Review:

The following properties apply to any real numbers a, b, m, n except where the expression is undefined.

- $a^0 = 1$
  (Example: $8^0 = 1$)
- $a^{-n} = \frac{1}{a^n}$
  (Example: $3^{-2} = \frac{1}{3^2}$)
- $\frac{1}{a^{-n}} = a^n$
  (Example: $\frac{1}{3^{-2}} = 3^2$)
- The Laws of Exponents:
  1. $a^m \cdot a^n = a^{m+n}$
     (Example: $4^2 \cdot 4^5 = 4^{2+5} = 4^7$)
  2. $\dfrac{a^m}{a^n} = a^{m-n}$
     (Example: $\frac{5^4}{5^3} = 5^{4-3} = 5$)
  3. $(a^m)^n = a^{mn}$
     (Example: $(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$)
- $(a \cdot b)^n = a^n \cdot b^n$
  (Example: $(2x)^3 = 2^3 x^3 = 8x^3$)
- $(a + b)^n \neq a^n + b^n$
  (Example: $(2 + x)^2$ IS NOT EQUAL TO $4 + x^2$)
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
  (Example: $\left(\dfrac{2}{y}\right)^2 = \dfrac{2^2}{y^2} = \dfrac{4}{y^2}$)
- $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n = \dfrac{b^n}{a^n}$
  (Example: $\left(\dfrac{2}{y}\right)^{-3} = \dfrac{y^3}{2^3} = \dfrac{y^3}{8}$)
- $a^{1/n} = \sqrt[n]{a}$
  (Example: $8^{1/3} = \sqrt[3]{8} = 2$)

- $a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$
  (Example: $4^{5/2} = \sqrt{4^5} = \sqrt{4}^5 = 2^5 = 32$)
- BE CAREFUL OF ORDER OF OPERATIONS!
- Remember: The Laws of Exponents only apply when the bases are the same!
- Remember: If n is an even integer and a is negative, $a^{1/n}$ is NOT a real number.

Function Composition

Definition. Let f and g be functions. Let $D(f)$ and $D(g)$ denote the domains of the functions f and g, respectively. Then we define the composition of f with g to be:

$$(f \circ g)(x) = f(g(x)),$$

(read: "f composed with g" or "f compose g"). The domain of $f \circ g$ is the set of all x in $D(g)$ such that $g(x)$ is in $D(f)$.

So, to find the domain of $f \circ g$, first start with the domain of g. Then throw out any values of x for which $g(x)$ is not in the domain of f.

Example 1. Our first example will be simple. Let $f(x) = 3x - 7$ and let $g(x) = 1 - x$. Note that the domain of both functions is all real numbers, $(-\infty, \infty)$. To find $f \circ g$, we do the following computation:

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{by definition} \\
&= f(1 - x) && \text{substitute } g(x) \\
&= 3(1 - x) - 7 && \text{plug into } f(x) \\
&= 3 - 3x - 7 && \text{distribute the 3} \\
&= -3x - 4 && \text{combine like terms}
\end{aligned}$$

So, the most simplified form of $f \circ g$ is $(f \circ g)(x) = -3x - 4$.

Now, to find the domain of $f \circ g$:

1. Start with the domain of g: $(-\infty, \infty)$
2. Find values of x where $g(x)$ is not in the domain of f.
    - The domain of f is all real numbers, and all the outputs of $g(x)$ are real numbers, so there is no output of $g(x)$ that is not in the domain of f.

3. Throw any values found in part 2 out of part 1 and that is the final answer.

   - We don't have anything to throw out from part 2, so we keep what we had in part 1: our final domain for $f \circ g$ is $(-\infty, \infty)$.

Example 2. Let $f(x) = \dfrac{x-2}{x-1}$ and let $g(x) = \dfrac{1}{x}$. Note that $D(f) = (-\infty, 1) \cup (1, \infty)$ and $D(g) = (-\infty, 0) \cup (0, \infty)$. To find $f \circ g$, we do the following computation:

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{by definition} \\
&= f\left(\frac{1}{x}\right) && \text{substitute } g(x) \\
&= \frac{\frac{1}{x} - 2}{\frac{1}{x} - 1} && \text{plug into } f(x) \\
&= \frac{\frac{1-2x}{x}}{\frac{1-x}{x}} && \text{add fractions on top and bottom using LCD} \\
&= \frac{1-2x}{x} \cdot \frac{x}{1-x} && \text{definition of dividing fractions} \\
&= \frac{(1-2x)x}{x(1-x)} && \text{definition of multiplying fractions} \\
&= \frac{1-2x}{1-x} && \text{cancel the common factor of } x \text{ on top and bottom}
\end{aligned}
$$

So, the most simplified form of $f \circ g$ is $(f \circ g)(x) = \frac{1-2x}{1-x}$.

Now, to find the domain of $f \circ g$:

1. Start with the domain of g: $(-\infty, 0) \cup (0, \infty)$

2. Find values of x where $g(x)$ is not in the domain of f.

   - The only value NOT in the domain of f is $x = 1$. So, we need to find values where $g(x) = 1$. $g(x) = \frac{1}{x} = 1$, solve for x. Then $x = 1$. So, the value of x for which $g(x)$ is NOT in the domain of f is $x = 1$.

3. Throw any values found in part 2 out of part 1 and that is the final answer.

    - We take $(-\infty, 0) \cup (0, \infty)$ and throw out $x = 1$. Our final domain for $f \circ g$ is $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

Rewriting functions as compositions of other functions

Example 3. Consider the function $H(x) = (x^2 - 3)^10$. Can we write H as a composition of two other functions? Note that we're raising something to the 10th power, so one of our functions can be $f(x) = x^{10}$. Then, what are we raising to the 10th power? It's $x^2 - 3$. This is what we're plugging into $f(x)$, so this is what we'll call $g(x)$. $g(x) = x^2 - 3$. Check it: is it true that $(f \circ g)(x) = H(x)$?

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 3) = (x^2 - 3)^{10} = H(x)$$

Example 4. Let $H(x) = \dfrac{1}{|x^3 - 1|}$. How can we decompose $H(x)$ into a composition of other, simpler functions? Here are some ways:

- $H(x) = (f \circ g)(x)$, where $f(x) = \frac{1}{|x|}$ and $g(x) = x^3 - 1$.
- $H(x) = (f \circ g)(x)$, where $f(x) = \frac{1}{x}$ and $g(x) = |x^3 - 1|$.
- $H(x) = (f \circ g \circ h)(x)$, where $f(x) = \frac{1}{x}$, $g(x) = |x|$, and $h(x) = x^3 - 1$.

Exercises

Simplify all answers completely. For problems 6-9, do not use $f(x) = x$ or $g(x) = x$ in your answer.

1. Compute $f \circ g$ and its domain: $f(x) = 2x - 1$; $g(x) = x^2$
2. Compute $f \circ g$ and its domain: $f(x) = x^3$; $g(x) = 2x - 3$
3. Compute $f \circ g$: $f(x) = x^{-2}$; $g(x) = \dfrac{5x-1}{x+1}$
4. Compute $f \circ g$: $f(x) = x^2 + 2x$; $g(x) = \sqrt{x+2}$
5. Compute $f \circ g$: $f(x) = \sqrt{x+2}$; $g(x) = x^2 + 2x$
6. Find two functions f and g such that $H(x) = (f \circ g)(x)$. $H(x) = \dfrac{5}{2x+3}$
7. Find two functions f and g such that $H(x) = (f \circ g)(x)$. $H(x) = (x-7)^4$
8. Find two functions f and g such that $H(x) = (f \circ g)(x)$. $H(x) = \frac{2}{3}(3x-2)^2$
9. Find two functions f and g such that $H(x) = (f \circ g)(x)$. $H(x) = \sqrt[3]{1+\sqrt{x}}$
10. Let $H(x) = 5x - 1$, $f(x) = \frac{1}{x}$, and $g(x) = \frac{1}{5x-1}$. Is the domain of $f \circ g$ equal to the domain of H? Why or why not?