Trigonometry Review Information

Contents

Basic Trigonometric Functions

There are three important ways to define and think about trig functions, especially sine and cosine:

1. Using the unit circle
2. Using the graph of the function
3. Using right triangles

Once we've used one of the above approaches to define $\sin\theta$ and $\cos\theta$, we can define four more trig functions in the following way:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

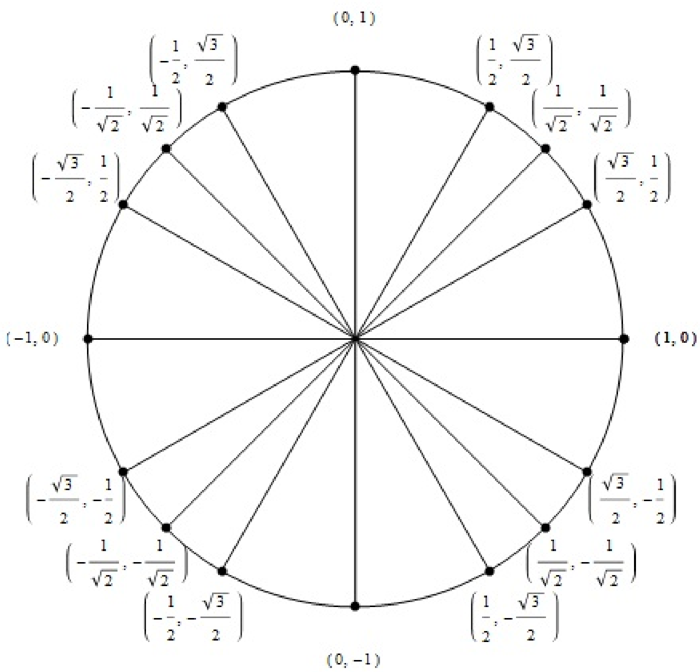
There are 7 important angles/traces to remember: $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}$ (or, 30, 60, and 45 in degrees) and $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ (or, 0, 90, 180, 270 in degrees).

1: The Unit Circle Approach

- Draw the unit circle (circle of radius 1 centered at the point (0,0)).
- Consider the point $(1, 0)$ - this is 0 radians/degrees.
- Measure positive angles θ (or, equivalently, traces t of the arc of the circle) in a counter clockwise direction (and negative angle/trace in clockwise direction). So, the positive x-axis is 90 degrees or $\frac{\pi}{2}$ radians.
- Mark the (x, y) coordinates of the points on the unit circle that have angle/trace equal to the 7 special values listed above. Also mark the (x, y) coordinates of any point on the unit circle where the measure of the smallest angle/shortest trace from that point along the unit circle to the x axis (called the *reference number* or *reference angle* is equal to one of the first 3 special values above.
- For a trace t which arrives at the point (x, y) on the unit circle, define sine and cosine of t by $\sin t = y$ and $\cos t = x$.

- For an angle θ which arrives at the point (x, y) on the unit circle, define sine and cosine of θ by $\sin\theta = y$ and $\cos\theta = x$.

- Note that $\sin t$ is positive in QI and QII and negative in QIII and QIV. Note that $\cos t$ is positive in QI and QIV and negative in QII and QIII.

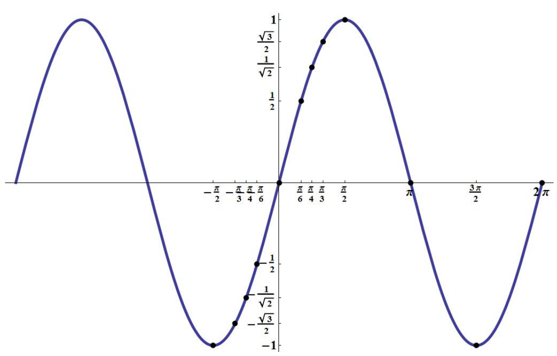


Some example values:

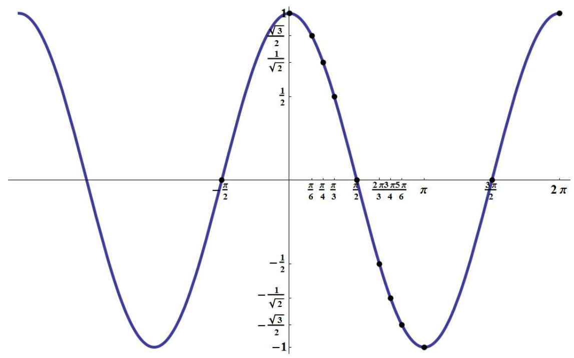
t	$\sin t$	$\cos t$	-	t	$\sin t$	$\cos t$
0	0	1	-	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	-	π	0	-1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	-	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1	0	-	$\frac{3\pi}{2}$	-1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	-	$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$

2: The Graphs of the Trig Functions

Once you've traced around the unit circle once, you can keep tracing (in both directions) and define sine and cosine for all real numbers. However, each time you complete one trip around the unit circle (every 2π radians), the values for sine and cosine will repeat. So, these functions are periodic, with period 2π. Approximately two periods of each function is shown below.

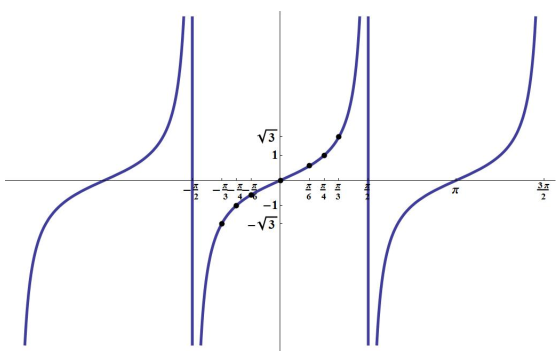


$y = \sin x$ on $[-2\pi, 2\pi]$

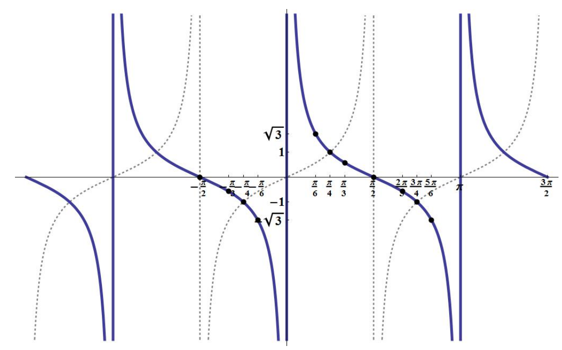


$y = \cos x$ on $[-2\pi, 2\pi]$

We can also look at the graphs of tangent, cotangent, cosecant, and secant. (The lines that appear to be vertical are, in fact, vertical asymptotes of the function.)

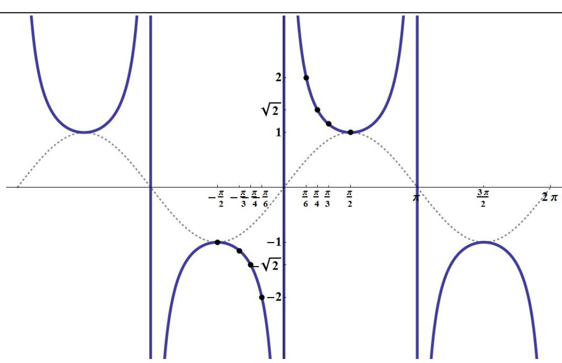


$y = \tan x$ on $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$



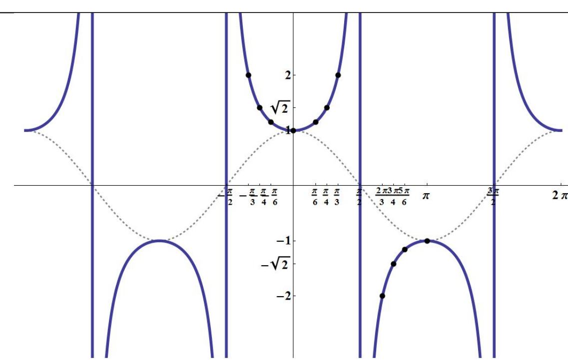
$y = \cot x$ on $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

(reciprocal of $\tan x$, shown in dashed gray)



$y = \csc x$ on $[-2\pi, 2\pi]$

(reciprocal of $\sin x$, shown in dashed gray)



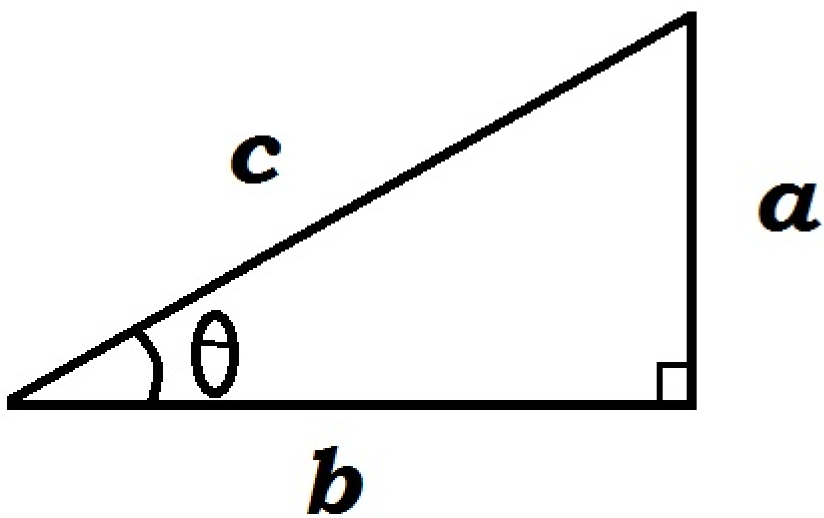
$y = \sec x$ on $[-2\pi, 2\pi]$

(reciprocal of $\cos x$, shown in dashed gray)

Note that QI corresponds to x values $0 < x < \frac{\pi}{2}$ on the graphs, QII corresponds to x values $\frac{\pi}{2} < x < \pi$, QIII corresponds to x values $\pi < x < \frac{3\pi}{2}$, and QIV corresponds to x values $\frac{3\pi}{2} < x < 2\pi$ (or any of the given intervals $+2k\pi$ for some integer π). If you have the graph memorized, you will be able to determine the sign of each trig function in each quadrant: the function is positive for all quadrants where the graph is above the x-axis and it is negative for all quadrants where the graph is below the x-axis.

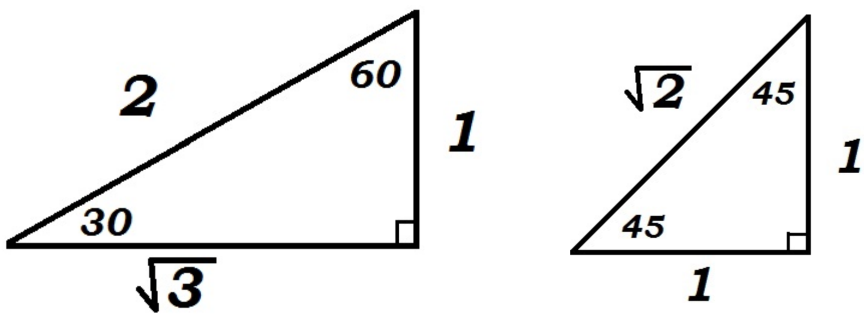
3: Trig and Right Triangles

If θ is an acute angle in a right triangle, we can define sine and cosine in terms of the lengths of the sides of the triangle as follows:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

In particular, we can consider the lengths of the sides of two special triangles:



For 30, 60, and 45 degree angles, we obtain the same values of sine and cosine as shown in the table using the unit circle approach.

The right triangle approach is no good on its own - you need the graph of the function or the unit circle to be able to analyze the values of the trig functions for angle/trace greater than or equal to 90 degrees ($\frac{\pi}{2}$ radians). But sometimes the triangle approach is helpful in other applications, including for inverse trig functions (the next section in this review) and for trig substitution integrals (which you will encounter in Calculus 2).

Inverse Trig Functions

Definition. Suppose $f(x)$ is a function. Define a relation $f^{-1}(x)$ by $f^{-1}(a) = b$ if and only if $f(b) = a$. If f^{-1} is a function, then we call it the inverse function of f.

Important General Facts about Inverse Functions:

- A function f has an inverse function if and only if f is one-to-one.
- The -1 exponent on f in this notation means inverse. It does NOT mean reciprocal, and the reciprocal of f is (usually) NOT equal to the inverse of f. That is,
$$\frac{1}{f(x)} \neq f^{-1}(x).$$
- The following identities hold for inverse functions: $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$. (If you compose f and f^{-1}, you always get back out whatever you put in, no matter which order you compose them in.)
- The domain of f is equal to the range of f^{-1} and the range of f is equal to the domain of f^{-1}.
- The graph of f^{-1} is equal to the graph of f reflected over the line $y = x$. Or, said another way, the graphs of f and f^{-1} are symmetric about the line $y = x$.

Note that NONE of the trig functions are one-to-one (they are all periodic with infinite domains and their graphs do not pass the horizontal line test), so NONE of them actually have inverse functions. But, what we can do is restrict the domain of each trig function to a piece of domain where it is one-to-one AND where it hits every point in its range. Then we can define an inverse function for this "restricted" trig function. Below is the summary of the standard restricted domains we choose for each trig function, and some facts about its inverse.

$f(x)$	Domain of f	Restricted Domain of f = Range of f^{-1}	Domain of f^{-1} = Range of f
$\sin x$	$(-\infty, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1, 1]$
$\cos x$	$(-\infty, \infty)$	$[0, \pi]$	$[-1, 1]$
$\tan x$	$\left\{x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + k\pi \text{ for } k \in \mathbb{Z}\right\}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(-\infty, \infty)$
$\csc x$	$\{x \in \mathbb{R} \mid x \neq k\pi \text{ for } k \in \mathbb{Z}\}$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$	$(-\infty, -1] \cup [1, \infty)$
$\sec x$	$\left\{x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + 2k\pi \text{ for } k \in \mathbb{Z}\right\}$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$	$(-\infty, -1] \cup [1, \infty)$
$\cot x$	$\{x \in \mathbb{R} \mid x \neq k\pi \text{ for } k \in \mathbb{Z}\}$	$(0, \pi)$	$(-\infty, \infty)$

Because the inverse trig functions are built from the original trig functions *with restricted domain*, we only get $(f \circ f^{-1})(x) = x$ is true for all x in the domain of the inverse trig function - where f is any of the six trig functions. (For example, $sin(sin^{-1}(x)) = x$ for any x in $[-1, 1]$.)

It is NOT true that $(f^{-1} \circ f)(x) = x$ for all x in the domain of f. This is only true for x that are in the RESTRICTED domain of f (or the range of f^{-1}). When x is outside this range, you need to think about the problem more carefully. (For example, $cos^{-1}(cos(\frac{\pi}{6})) = \frac{\pi}{6}$ because the restricted domain of cosine is $[0, \pi]$ and $\frac{\pi}{6}$ is in that interval. On the other hand, $cos^{-1}(cos(\frac{5\pi}{4})) = \frac{3\pi}{4} \neq \frac{5\pi}{4}$ because $\frac{5\pi}{4}$ is not in the interval $[0, \pi]$.)

If you need to find $f^{-1}(x)$ for some x in the domain of an inverse trig function, you should think to yourself: "What is the angle θ *in the restricted domain of* f that has $f(\theta) = x$?" For example, if I want to find $\tan^{-1}(-1)$, I think to myself: "What is the angle θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (ie, in the restricted domain of $\tan x$) that has tangent equal to -1 (ie, where is $\tan\theta = -1$)?" Then I realize the answer must be $-\frac{\pi}{4}$. The way the inverse functions are built, there can only ever be one answer - you just have to pick the one in the right range (in the restricted domain of the trig function).