THEOREM 0 (*The Relationship Between One-sided and Two-sided Limits*)

$$\lim_{x \to a} f(x) = L \quad \Longleftrightarrow \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Some basic tools for finding limits algebraically.

THEOREM 1.2.2 (*Properties of Limits*)

Let a be a real number and suppose that

$$\lim_{x \to a} f(x) = L_1 \quad \text{and} \quad \lim_{x \to a} g(x) = L_2$$

That is, both limits exist and have values L_1 and L_2, respectively. Then:

(a) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$

(b) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L_1 - L_2$

(c) $\lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = L_1 L_2$

(d) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L_1}{L_2}$, provided $L_2 \neq 0$

(e) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L_1}$, provided $L_1 > 0$ if n is even

You read these as:

(a) *The limit of a sum is the sum of the limits.*

(b) *The limit of a difference is the difference of the limits.*

(c) *The limit of a product is the product of the limits.*

(d) *The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero.*

(e) *The limit of an nth root is the nth root of the limit.*

THEOREM 1.2.4 (*Limit of a Rational Function*)

Let

$$f(x) = \frac{p(x)}{q(x)}$$

be a rational function, and let a be any real number.

(a) If $q(a) \neq 0$, then $\lim_{x \to a} f(x) = f(a)$.

(b) If $q(a) = 0$, but $p(a) \neq 0$, then $\lim_{x \to a} f(x)$ *does not exist.*

DEFINITION 1.3.1 (*Limits at Infinity*)

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

DEFINITION 1.3.1a (*Horizontal Asymptote*)

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

We have the same limit properties we saw last time, but now we begin with

$$\lim_{x\to\infty} f(x) = L_1 \quad \text{and} \quad \lim_{x\to\infty} g(x) = L_2$$

We can add the following properties,

THEOREM (*Properties of Limits at Infinity*)

(a) $\lim_{x\to\infty} (f(x))^n = \left(\lim_{x\to\infty} f(x)\right)^n$

(b) $\lim_{x\to\infty} (kf(x)) = k \lim_{x\to\infty} f(x)$

As well as several *obvious* limits:

1. $\lim_{x\to\infty} x = \infty$
2. $\lim_{x\to\infty} x^n = \infty, \quad n = 1, 2, 3, \ldots$
3. $\lim_{x\to\infty} \frac{1}{x} = 0$
4. $\lim_{x\to\infty} \frac{1}{x^n} = 0, \quad n = 1, 2, 3, \ldots$

DEFINITION 1.5.1 (*Continuity*)

A function f is said to be **continuous** at $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

This requires the following three conditions be satisfied:

1. $f(c)$ is defined (that is, c is in the domain of f).
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

THEOREM 1.5.4

(a) A polynomial is continuous everywhere.

(b) A rational function is continuous st every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero. In other words it is continuous on its domain.

THEOREM 1.5.5 (*Continuity of Compositions*)

If f is continuous at L and $\lim_{x \to c} g(x) = L$, then $\lim_{x \to c} f(g(x)) = f(L)$. That is,

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$$

Theorem 1.5.5 says we can drag limits through continuous functions.

THEOREM 1.5.7 (*The Intermediate Value Theorem*)

If f is continuous on a closed interval $[a, b]$ and N is any number between $f(a)$ and $f(b)$, inclusive, then there exists a number c in the interval $[a, b]$ such that $f(c) = N$.

One procedure for approximating roots is based on this consequence of the I.V.T.

THEOREM 1.5.8

If f is continuous on $[a, b]$, and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).

THEOREM 1.6.5

$$\text{(a)} \lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{(b)} \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

THEOREM

If $f(x) \leq g(x)$ when x is near c (except possibly at c) and the limits of f and g both exist as x approaches c then

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$$

THEOREM 1.6.4 (*The Squeeze Theorem*)

Let f, g, and h be functions satisfying

$$g(x) \leq f(x) \leq h(x)$$

for all x in some open interval containing the number c, (except possibly at c). If

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then

$$\lim_{x\to c} f(x) = L$$