BEFORE You Begin Calculus II

If it has been awhile since you had Calculus 1, I strongly suggest that you refresh both your differentiation and integration skills. I would also like to remind you that in Calculus 2, as a *mathematics* course, connecting your work and correct notation are part of any correct solution. So when you review, always practice in the manner that you will deliver on exams, that is, *write mathematics*.

Differentiation

You are required to **know** the differentiation formulas from calculus I (in the table below), as well as the product rule, quotient rule, chain rule, and the various properties of derivatives. You will not have a formula sheet available on quizzes or exams with these formulas.

For derivative practice, go to my web-page, *math.mercyhurst.edu/∼griff/courses/m170/HW.php*; you should work, as necessary, the exercises from sections: `2.3, 2.4, 2.5, 2.6, 3.2, 3.3` in the text *Calculus - Early Transcendentals* by Anton (product and quotient rules, chain rule, derivatives of logarithmic, exponential, and inverse trigonometric functions).

You should also review L'Hôpital's Rule, section 3.6; follow the homework link above for exercises.

For further practice, I might suggest also working problems (as necessary) from the attached 'Derivative Problems' worksheet. If you wish to see more than the provided answers, follow the link: Derivative Problems, the file *SOLN_derivatives.pdf* which has scanned copies of the worked solutions.

Integration

You are required to **know** both the derivative and integration formulas from calculus I (in the table below), the method of substitution for indefinite and definite integrals (using correct notation), the Fundamental Theorem of Calculus (Pt 1), along with the various properties of integrals.

You should work, as necessary, the exercises from sections: `5.2, 5.3, (5.5 & 5.6` light), `5.9` (complete); from the HW.

Differentiation Formulas	Integration Formulas
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r$	$\int x^r \; dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x \; dx = \sin x + C$
4. $\frac{d}{dx}[\cos x] = -\sin x$	$\int -\sin x \; dx = \cos x + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x \; dx = \tan x + C$
6. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x \; dx = \sec x + C$
7. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$	$\int \frac{1}{u} \; du = \ln\lvert u\rvert + C$
8. $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$	$\int e^u \; du = e^u + C$
9. $\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$	$\int \frac{1}{\sqrt{a^2-u^2}} \; du = \sin^{-1}\frac{u}{a} + C$
10. $\frac{d}{dx}\left[\frac{1}{a}\tan^{-1}\frac{u}{a}\right] = \frac{1}{a^2+u^2}\frac{du}{dx}$	$\int \frac{1}{a^2+u^2} \; du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

What is the concept "Limit"?

It is reasonable to claim that the foundation for all of calculus is indeed the concept "limit".

Our notation for a limit:

$$\lim_{x \to c} f(x) = L$$

The Relationship Between One-sided and Two-sided Limits:

$$\lim_{x \to c} f(x) = L \quad \Longleftrightarrow \quad \lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$$

Limits we Need:

- $\lim_{x \to c} k = k$
- $\lim_{x \to c} x = c$
- $\lim_{x \to \infty} x = \infty$ or $\lim_{x \to \infty} x^n = \infty$ for $n = 1, 2, 3, 4, \ldots$
- $\lim_{x \to \infty} \frac{1}{x} = 0$ or $\lim_{x \to \infty} \frac{1}{x^n} = 0$ for $n = 1, 2, 3, 4, \ldots$

Properties:

Let c be any real number, k a constant, and suppose f, g have limits that exist at c. Then:

- *You can drag constants through limits*
  $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$
- *The limit of a Sum/difference*
  $\lim_{x \to c} \big[f(x) \pm g(x)\big] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
- *The limit of a Product*
  $\lim_{x \to c} f(x)g(x) = \Big(\lim_{x \to c} f(x)\Big)\Big(\lim_{x \to c} g(x)\Big)$
- *The limit of a Quotient*
  $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}, \quad \text{if } \lim_{x \to c} g(x) \neq 0$
- *The limit of a Power - generalized below*
  $\lim_{x \to c} \big[f(x)\big]^r = \Big[\lim_{x \to c} f(x)\Big]^r$
  where r is a real number. If r is negative, then $\lim_{x \to c} f(x) \neq 0$.

An important Theorem:
If $\lim_{x \to c} g(x) = L$ and if the function f is continuous at L, then

$$\lim_{x \to c} f(g(x)) \quad = \quad f\Big(\lim_{x \to c} g(x)\Big)$$

This states that limits can be dragged through continuous functions.

The Squeeze Theorem
Let f, g and h be functions satisfying

$$g(x) \leq f(x) \leq h(x)$$

for all x in some open interval containing the number c, except possibly at c. If

$$\lim_{x \to c} g(x) \;=\; \lim_{x \to c} h(x) \;=\; L$$

then f has the same limit,

$$\lim_{x \to c} f(x) \;=\; L$$

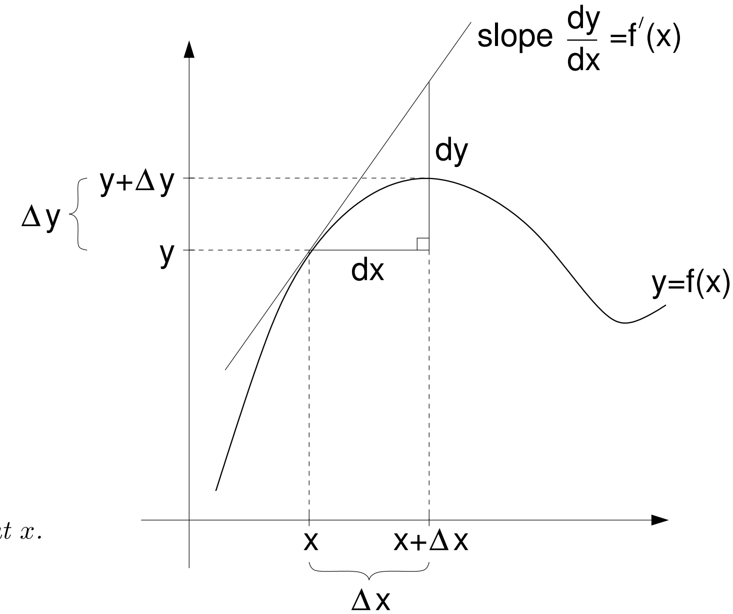
What is the average change?

The **average rate of change** of a function $f(x)$ between two values $x = a$ and $x = b$ is given by the quotient $(f(b) - f(a))/(b - a)$. Geometrically, the average change is the slope of the (secant) line passing through the points $(a, f(a))$ and $(b, f(b))$. If one uses the points $(x, f(x))$ and $(x+h, f(x+h))$, where h is an arbitrary constant, the expression for the average change becomes

$$\text{average change} = \frac{f(x+h) - f(x)}{h}.$$

Alternatively, one can write the average change $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta y}$.

- ⋄ The limit of difference quotients.
- ⋄ The instantaneous rate of change of a function.
- ⋄ The slope of a line tangent to the graph of a function.
- ⋄ The "best" linear approximation to a function.
- ⋄ Various rules and tables for computing.



The **derivative** $f'(x)$ is *the instantaneous rate of change of* f *at* x. Geometrically, the derivative is the slope of the line tangent to the graph of $y = f(x)$ (see the figure).

For a function $y = f(x)$, we take as the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Letting $h = \Delta x$, the definition can be written

$$\frac{dy}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The expressions dx and dy are called **differentials** and are related to Δx and Δy as follows.

$$\Delta x \equiv \text{the change in } x, \text{ and } \Delta x = \mathrm{d}x$$
$$\Delta y = \text{the change in } y = f(x + \Delta x) - f(x)$$
$$dy = f'(x)\Delta x$$

In general, $dy \neq \Delta y$.

The sign of the derivative determines whether a function is increasing, decreasing, or neither.

> If $f'(x) > 0$ then f (the y value) is increasing, and if $f'(x) < 0$ then f is decreasing. If $f'(x) = 0$ then f is neither increasing or decreasing.

What is the integral?

- ◇ The limit of Riemann sums.
- ◇ The area under the graph of a function.
- ◇ An antiderivative.
- ◇ Various rules and tables for computing.

A function $F(x)$ is an **antiderivative** of the function $f(x)$ on an interval I, if $F'(x) = f(x)$ for all x in I. The **indefinite integral**[1] $\int f(x)\ \mathrm{d}x$ is *the antiderivative of f with respect to x*, that is,

$$\int f(x)\ \mathrm{d}x = F(x) + C$$

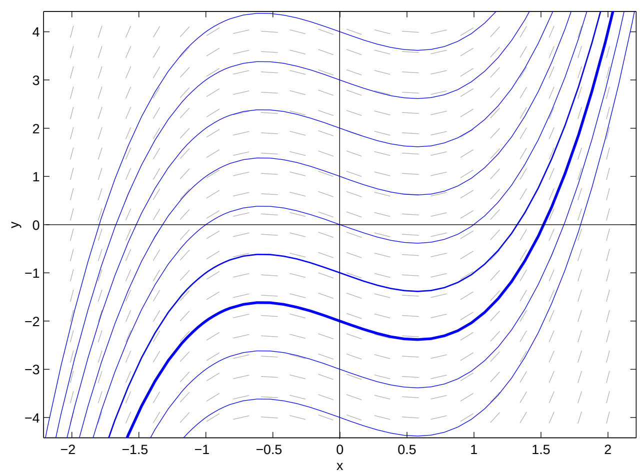
where C is an arbitrary constant called the **constant of integration**. Integration is the "inverse" of differentiation, by substituting $F'(x)$ for $f(x)$ we obtain

$$\int F'(x)\ \mathrm{d}x = F(x) + C.$$

Moreover, if $\int f(x)\ \mathrm{d}x = F(x) + C$, then

$$\frac{d}{dx}\left[\int f(x)\ \mathrm{d}x\right] = f(x).$$

Geometrically, the integral is the collection of graphs $F(x) + C$ whose tangent lines have slopes given by $f(x)$.



The figure shows the graphs of several antiderivatives of $dy/dx = 3x^2 - 1$, that is, the graphs $y = \int (3x^2 - 1)\ \mathrm{d}x = x^3 - x + C$ for various values of C. The expression $x^3 - x + C$ is called the general antiderivative of $3x^2 - 1$. The particular antiderivative that satisfies the condition $y(2) = 4$ is $y = x^3 - x - 2$ (the bold curve in the figure).

The integral $\int_a^b f(x)\ \mathrm{d}x$ is called a **definite integral** and computes the "net signed area" between f, the x-axis, and the lines $x = a$ and $x = b$.

The Fundamental Theorem of Calculus (Part I): If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$,

$$\int_a^b f(x)\ \mathrm{d}x\ =\ F(b) - F(a)$$

Integration by Substitution Suppose that F is an antiderivative of f and that g is a differentiable function. To integrate

$$\int f(g(x))g'(x)\ \mathrm{d}x \qquad (1)$$

we make the substitution

$$u = g(x) \quad \text{and} \quad \mathrm{d}u = g'(x)\,\mathrm{d}x$$

so that eq **??** becomes:

$$\int f(u)\ \mathrm{d}u\ =\ F(u) + C$$

[1]The term **indefinite integral** is a synonym for antiderivative.

Derivative Problems

Find the derivative of each of the given functions.

1. $y = (x^2 + 4x + 6)^5$
2. $f(t) = \dfrac{1}{(t^2 - 2t - 5)^4}$
3. $h(t) = \left(t - \dfrac{1}{t}\right)^{3/2}$
4. $y = \dfrac{1}{\sqrt[5]{x^2}}$
5. $G(x) = (3x - 2)^{10}(5x^2 - x + 1)^{12}$
6. $y = (2x - 4)^4(8x^2 - 4)^{-3}$
7. $y = (x^2 + 1)\sqrt[3]{x^2 + 2}$
8. $y = \sec^2 x + \tan^2 x$
9. $R(y) = \dfrac{y^2 - 1}{(3y + 1)^2}$
10. $f(x) = \left(3x^2\right)(4x)^{1/2}$
11. $g(x) = \dfrac{2}{x^4 - x^2 + 1}$
12. $h(x) = \sqrt[5]{(3x^2 - 2x)^4}$
13. $f(x) = (\sin x \ \sin 3x)^9$
14. $y = \dfrac{1 + \sin x}{x + \cos x}$
15. $f(x) = \dfrac{\cos x}{\sin x}(\sin x + \tan x)$
16. $g(x) = \dfrac{2x^4 + 3x^2 - 1}{x^2}$
17. $s(t) = \sqrt[4]{\dfrac{t^3 + 1}{t^3 - 1}}$
18. $f(z) = \dfrac{1}{\sqrt[5]{2z - 1}}$
19. $h(x) = \dfrac{x}{\sqrt{7 - 3x}}$
20. $f(x) = \left(2x^{3/4} + 5x^{-1/6}\right)^{12}$
21. $y = \dfrac{\cos\left(a^3 + x^3\right)}{3}$
22. $s = \left(\dfrac{1 + t^2}{1 - t^2}\right)^7$
23. $y = \dfrac{\tan x - 1}{\sec x}$
24. $f(x) = \dfrac{\tan^2 x}{\sqrt{\sin^6 x + \sin^4 x \cos^2 x}}$
25. $h(x) = \left(x^2 + (x^2 + 9)^{1/2}\right)^{1/2}$
26. $r(t) = \sqrt[3]{\dfrac{2t + 5}{7t - 2}}$
27. $y = \sqrt{x + \sqrt{x}}$
28. $R = \dfrac{\sqrt{t} + 1}{\sqrt{t} - 1}$
29. $f(x) = \left(\dfrac{\cos\left(x^2\right)\tan^2\left(x^2\right)}{\sec\left(x^2\right)}\right)^3$
30. $y = \dfrac{2x}{(3x^2 - 4)^{1/3}}$
31. $y = e^{k \tan \sqrt{x}}$
32. $y = (\ln \tan x)^2$
33. $y = \sqrt{1 + 2xe^{-2x}}$
34. $f(x) = \dfrac{1}{1 + \ln x}$
35. $y = \cos\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$
36. $r(\theta) = \tan^{-1}(\cos\theta)$
37. $f(t) = \sin^2\left(e^{\sin^2 t}\right)$
38. $y = \tan^{-1}\left(x - \sqrt{x^2 + 1}\right)$

Find the **second** derivative of each of the given functions.

39. $g(x) = \dfrac{2x+1}{x-1}$

40. $f(y) = \dfrac{y}{\sqrt{1-y^2}}$

41. $h(x) = \dfrac{3x}{\sqrt{2x^2+7}}$

42. $y = \dfrac{3}{(5-2x^2)^{3/4}}$

43. If f and g are differentiable functions such that $f(2) = 3, f'(2) = -1, g(2) = -5$, and $g'(2) = 2$, find the following values.

(a) $(f+g)'(2)$

(b) $(4f)'(2)$

(c) $(fg)'(2)$

(d) $(ff)'(2)$

(e) $\left(\frac{1}{f+g}\right)'(2)$

(f) $\left(\frac{5}{g}\right)'(2)$

44. Given the following table of values, find the indicated derivatives.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	4	2	6
2	1	5	8	7
3	7	7	2	9

(a) $h'(1)$, where $h(x) = f(g(x))$

(b) $H'(1)$, where $H(x) = g(f(x))$

(c) $F'(2)$, where $F(x) = f(f(x))$

(d) $G'(3)$, where $G(x) = g(g(x))$

Solutions:

1. $y' = 10(x^2+4x+6)^4(x+2)$

2. $f'(t) = \dfrac{8(1-t)}{(t^2-2t-5)^5}$

3. $h'(t) = \dfrac{3}{2}\left(t-\dfrac{1}{t}\right)^{1/2}\left(1+\dfrac{1}{t^2}\right)$

4. $\dfrac{dy}{dx} = \dfrac{-2}{5\sqrt[5]{x^7}}$

5. $G'(x) = 30(3x-2)^9(5x^2-x+1)^{12} + 12(10x-1)(3x-2)^{10}(5x^2-x+1)^{11}$
$= 6(3x-2)^9(5x^2-x+1)^{11}(85x^2-51x+9)$

6. $y' = 8(2x-4)^3(8x^2-4)^{-3} - 48x(2x-4)^4(8x^2-4)^{-4}$

7. $\dfrac{dy}{dx} = 2x\sqrt[3]{x^2+2} + \left(\dfrac{2x}{3}\right)(x^2+1)(x^2+2)^{-2/3}$

8. $\dfrac{dy}{dx} = 4\sec^2 x \tan x$

9. $R'(y) = \dfrac{2(y+3)}{(3y+1)^3}$

10. $f'(x) = 15x^{3/2}$

11. $g'(x) = -2(x^4-x^2+1)^{-2}(4x^3-2x)$

12. $h'(x) = \dfrac{8(3x-1)}{5(3x^2-2x)^{1/5}}$

13. $f'(x) = 9(\sin x\ \sin 3x)^8(\cos x \sin 3x + 3\sin x \cos 3x)$

14. $\dfrac{dy}{dx} = \dfrac{x\cos x}{(x+\cos x)^2}$

15. $f'(x) = -\sin x$

16. $g'(x) = 4x + 2x^{-3}$

17. $s'(t) = -\dfrac{1}{2}\left(\dfrac{t^3+1}{t^3-1}\right)^{-3/4}\dfrac{3t^2}{(t^3-1)^2}$

18. $f'(z) = -\dfrac{2}{5}(2z-1)^{-6/5}$

19. $h'(x) = \dfrac{14-3x}{2(7-3x)^{3/2}}$

20. $f'(x) = 12\left(2x^{3/4}+5x^{-1/6}\right)^{11}\left(\dfrac{3}{2}x^{-1/4} - \dfrac{5}{6}x^{-7/6}\right)$

21. $\dfrac{dy}{dx} = -x^2\sin(a^3+x^3)$

22. $\dfrac{ds}{dt} = 7\left(\dfrac{1+t^2}{1-t^2}\right)^6\dfrac{4t}{(1-t^2)^2}$

23. $\dfrac{dy}{dx} = \cos x + \sin x$

24. $f'(x) = 2\sec^2 x \tan x$

25. $h'(x) = \dfrac{1}{2}\left(x^2+(x^2+9)^{1/2}\right)^{-1/2}(2x + x(x^2+9)^{-1/2})$

26. $r'(t) = \left(\dfrac{2t+5}{7t-2}\right)^{-2/3}\dfrac{-13}{(7t-2)^2}$

27. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\dfrac{1}{2\sqrt{x}}\right)$

28. $\dfrac{dR}{dt} = \dfrac{-1}{\sqrt{t}(\sqrt{t}-1)^2}$

29. $f'(x) = 12x\ \sin^5(x^2)\cos(x^2)$

30. $\dfrac{dy}{dx} = \dfrac{2x^2-8}{(3x^2-4)^{4/3}}$

31. $\dfrac{dy}{dx} = \dfrac{k}{2\sqrt{x}}\ \sec^2(\sqrt{x})\ e^{k\tan\sqrt{x}}$

32. $\dfrac{dy}{dx} = \dfrac{2\sec^2(x)\ \ln(\tan x)}{\tan x}$

33. $\dfrac{dy}{dx} = \dfrac{1-2x}{e^{2x}\sqrt{1+2xe^{-2x}}}$

34. $f'(x) = -\dfrac{1}{x(1+\ln x)^2}$

35. $\dfrac{dy}{dx} = \dfrac{4e^{2x}}{(1+e^{2x})^2}\ \sin\left(\dfrac{1-e^{2x}}{1+e^{2x}}\right)$

36. $r'(\theta) = -\dfrac{\sin\theta}{1+\cos^2\theta}$

37. $f'(t) = \sin(2t)\ \sin\left(2e^{\sin^2 t}\right)e^{\sin^2 t}$

38. $\dfrac{dy}{dx} = \dfrac{1}{2(x^2+1)}$

(37) from: $4\sin t\cos t\ \sin\left(e^{\sin^2 t}\right)\cos\left(e^{\sin^2 t}\right)\ e^{\sin^2 t}$

(38) from: $\dfrac{1-\frac{x}{\sqrt{x^2+1}}}{1+(x-\sqrt{x^2+1})^2}$

39. $g''(x) = 6(x-1)^{-3}$

40. $f''(y) = 3y(1-y^2)^{-5/2}$

41. $h''(x) = -126x(2x^2+7)^{-5/2}$

42. $\dfrac{d^2y}{dx^2} = \dfrac{45(1+x^2)}{(5-2x^2)^{11/4}}$

43. (a) 1 (b) -4
(c) 11 (d) -6
(e) $-\frac{1}{4}$ (f) $-\frac{2}{5}$

44. (a) $h'(1) = 30$ (b) $H'(1) = 36$
(c) $F'(2) = 20$ (d) $G'(3) = 63$