Euclid's The Elements

We know surprisingly little about the life of "Father of Geometry," except that he was a Greek mathematician who lived from c. 325 BC to c. 270 BC. What we do know is taken from the writings of his contemporaries, such as Archimedes. However, his mathematical work continues to define the field of geometry to this day. His 13 volume book *The Elements* is often considered to be the most influential textbook ever written. The number of printed editions of the *The Elements* is second only to the Bible.

It is not only the content that has importance, but the structure. Euclid's approach was the first of its kind in mathematics, and has set a standard still followed by mathematicians today. Euclid begins by painstaking defining each term he required throughout the text. Next, he included five postulates, or axioms. These are facts the he held to be true without proof. The remaining volumes of the book are propositions and their proofs, each building from the definitions, postulates, and earlier theorems. Many of these propositions state the possible figures we can construct, and their proofs serve as directions for these constructions.

The definitions, postulates, common notions, and propositions are included here, along with two sample propositions and their proofs.

The Definitions

1. A point is that which has no part.
2. A line is breadthless length.
3. The ends of a line are points.
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The edges of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.
8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called rectilinear.
10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.
11. An obtuse angle is an angle greater than a right angle.
12. An acute angle is an angle less than a right angle.
13. A boundary is that which is an extremity of anything.
14. A figure is that which is contained by any boundary or boundaries.
15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.
16. And the point is called the center of the circle.
17. A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.
18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.
19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.
20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.
21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.
22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.
23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

The Five Postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

The Five Common Notions

1. Things which equal the same thing also equal one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another equal one another.
5. The whole is greater than the part.

Note: The common notions, presented as axioms not intended to be proven, refer to magnitudes. They state that we can only compare or combine magnitudes when things are alike; for instance, we can join two rectangles together to make a larger one, but we cannot combine an angle and a triangle.

The Propositions

1. To construct an equilateral triangle on a given finite straight line.
2. To place a straight line equal to a given straight line with one end at a given point.
3. To cut off from the greater of two given unequal straight lines a straight line equal to the less.
4. If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.
5. In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.
6. If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.
7. Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.
8. If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.
9. To bisect a given rectilinear angle.
10. To bisect a given finite straight line.
11. To draw a straight line at right angles to a given straight line from a given point on it.
12. To draw a straight line perpendicular to a given infinite straight line from a given point not on it.
13. If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.
14. If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.
15. If two straight lines cut one another, then they make the vertical angles equal to one another.
16. In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.
17. In any triangle the sum of any two angles is less than two right angles.
18. In any triangle the angle opposite the greater side is greater.
19. In any triangle the side opposite the greater angle is greater.
20. In any triangle the sum of any two sides is greater than the remaining one.
21. If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

22. To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.
23. To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.
24. If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.
25. If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.
26. If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.
27. If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.
28. If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.
29. A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.
30. Straight lines parallel to the same straight line are also parallel to one another.
31. To draw a straight line through a given point parallel to a given straight line.
32. In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.
33. Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.
34. In parallelogrammic areas the opposite sides and angles equal one another, and the diameter bisects the areas.
35. Parallelograms which are on the same base and in the same parallels equal one another.
36. Parallelograms which are on equal bases and in the same parallels equal one another.
37. Triangles which are on the same base and in the same parallels equal one another.
38. Triangles which are on equal bases and in the same parallels equal one another.
39. Equal triangles which are on the same base and on the same side are also in the same parallels.
40. Equal triangles which are on equal bases and on the same side are also in the same parallels.
41. If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.
42. To construct a parallelogram equal to a given triangle in a given rectilinear angle.
43. In any parallelogram the complements of the parallelograms about the diameter equal one another.
44. To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle.
45. To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.
46. To describe a square on a given straight line.
47. In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.
48. If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.

Euclid's Other Works

Euclid also wrote texts on spherical geometry, number theory, logic, and rigor, as well as astronomy and optics. Many of his works are lost, but were mentioned by other mathematicians of the era.

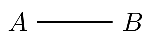
Proposition 1

To construct an equilateral triangle on a given finite straight line.

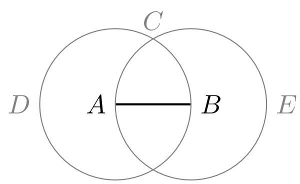
Note: The first proposition is special, as it can rely only on the definitions, postulates, and common notions.

Proof:

Let AB be the given finite straight line. It is required to construct an equilateral triangle on the straight line AB.



Describe the circle BCD with center A and radius AB. Again describe the circle ACE with center B and radius BA. (Postulate 3)



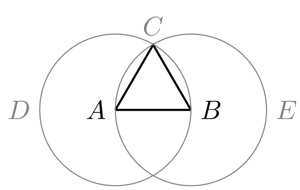
Join the straight lines CA and CB from the point C at which the circles cut one another to the points A and B. (Postulate 1)

Now, since the point A is the center of the circle CDB, therefore AC equals AB. Again, since the point B is the center of the circle CAE, therefore BC equals BA. (Definition 15)

But AC was proved equal to AB, therefore each of the straight lines AC and BC equals AB. (Common notion 1)

And things which equal the same thing also equal one another, therefore AC also equals BC. (Common notion 1)

Therefore the three straight lines AC, AB, and BC equal one another. (Common notion 1)



Therefore the triangle ABC is equilateral, and it has been constructed on the given finite straight line AB. (Definition 20) QED

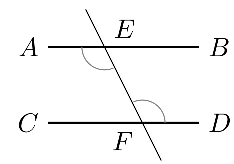
Proposition 27

If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

Note: This is an example of "proof by contradiction," which involves showing that assuming the intended conclusion is incorrect leads to other false statements.

Proof:

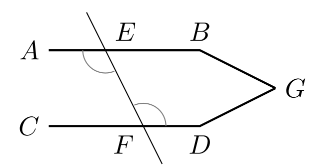
Let the straight line EF falling on the two straight lines AB and CD make the alternate angles AEF and EFD equal to one another.



I say that AB is parallel to CD.

If not, AB and CD when produced meet either in the direction of B and D or towards A and C.

Let them be produced and meet, in the direction of B and D, at G.



Then, in the triangle GEF, the exterior angle AEF equals the interior and opposite angle EFG, which is impossible. (Prop 16)

Therefore AB and CD when produced do not meet in the direction of B and D. Similarly it can be proved that neither do they meet towards A and C.

But straight lines which do not meet in either direction are parallel. Therefore AB is parallel to CD. (Definition 23) QED