What is Linear Algebra?

Linear algebra is a branch of mathematics that studies objects called vector spaces, their properties, and maps between them. It is an important area for anyone interested in mathematics, programming, physics, data science, and engineering, and even has applications in computer graphics. This course will start off with more traditional problems, such as finding solutions to systems of linear equations. Later, we'll see some more theoretical concepts, while still connecting them to real life applications.

Some of the topics we'll cover are listed below. Don't worry if you don't know what something means yet - I would not expect you to be familiar with any of these terms or concepts before the class begins.

- Vectors and vector operations: addition, the dot product, norm
- Matrices and matrix operations: addition, multiplication, determinants, inverses
- Solving systems of linear equations using Gaussian Elimination and matrix inverses
- Geometric interpretation of systems of linear equations
- Vector spaces and their properties
- Subspaces, bases, and spanning sets
- Linear transformations and their properties
- Eigenvalues, eigenvectors, and eigenspaces

What Should I Know Before Class Starts?

The official prerequisite for this course is Calculus I, though we will see very little calculus in the course. Derivatives may be mentioned, but you will not be required to integrate or solve calculus problems like related rates or optimization.

You should have a strong background in basic algebra, including the topics covered in this packet. Properties of real numbers, working with exponents, and functions will be referred to regularly in class, and it will be expected that you are familiar with these principles and definitions.

The notes in the packet are not meant to be thorough enough to learn from, but just to remind you of some topics you'll need to recognize throughout the course. If anything is completely unfamiliar, please ask me for suggested resources. Any precalculus text will cover most of the material, and there are many excellent online (free) resources as well.

About the Lab

This is a four credit course with a lab component. One day per week, we will meet in the computer lab to explore applications of linear algebra, letting the computer solve problems that would be unreasonable to approach by hand. While some programming will be involved, you are not expected to have any experience in programming.

Any software we use in class will be open source or at least free, so you will not need to invest in any materials besides the course textbook. However, if you would like to use the software outside of class, you will be able to install or access it on your home computer as well.

You will not need to have a computer in class or at home in order to complete lab assignments.

Required Materials

You will need to purchase or rent a copy of the textbook:
Linear Algebra and Its Applications, by David Lay, Steven Lay, and Judi McDonald, 5th Edition

This textbook may be available for purchase bundled with software or additional manuals. *You only need to purchase the textbook for this course*. No other resources are required. Be sure to check the edition when ordering.

ALGEBRAIC PROPERTIES

- Associative Property:

For Addition: $a + (b + c) = (a + b) + c$
For Multiplication: $a(bc) = (ab)c$

- Commutative Property:

For Addition: $a + b = b + a$
For Multiplication: $ab = ba$

- Distributive Property:

$$a(b + c) = ab + ac$$

- Zero Product Principle:

If $ab = 0$, then either $a = 0$ or $b = 0$

PROPERTIES OF EXPONENTS

$$a^n a^m = a^{n+m} \qquad \frac{a^n}{a^m} = a^{n-m} \qquad a^n b^n = (ab)^n \qquad a^0 = 1 \qquad a^{-1} = \frac{1}{a} \qquad a^{1/n} = \sqrt[n]{a}$$

FUNCTIONS

- $f(x)$ is a **function** if it has exactly one value for each x in its domain.

$f(x) = x^{1/2}$ is not a function because $f(9)$ could be -3 or 3

- A function must pass the vertical line test: a vertical line drawn through the graph of the function cannot intersect the graph more than once.
- The **domain** of a function $f(x)$ is the set of all values of x for which f is defined.
- The **range** of a function is the set of all values that $f(x)$ can take over all x in its domain.
- Composition of functions:

$$(f \circ g)(x) = f(g(x))$$

- Function composition is not commutative; in general $(f \circ g)(x) \neq (g \circ f)(x)$
- A function $f(x)$ is **one-to-one** (aka injective) if whenever $f(x_1) = f(x_2)$ we have $x_1 = x_2$

$f(x) = x^2$ is not a one-to-one because $f(2) = 4$ and $f(-2) = 4$ but $2 \neq -2$

- A one-to-one function must pass the horizontal line test: a horizontal line drawn through the graph of the function cannot intersect the graph more than once.
- A function $f(x)$ is **invertible** if there exists another function $f^{-1}(x)$ such that

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$$

- A function is invertible if and only if it is one-to-one.

TYPES OF NUMBERS

- A **natural number** is a non-negative, whole number: $0, 1, 2, 3, 4, 5, \cdots$. Denoted $\mathbb{N}$.
- An **integer** is a whole number: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. They include the natural numbers. Denoted $\mathbb{Z}$.
- A **rational number** can be written as a quotient $\frac{a}{b}$ where a and b have no common factors. They include the integers. Denoted $\mathbb{Q}$.
- An **irrational number** is a number that cannot be written as a quotient $\frac{a}{b}$.
- A **real number** is any number found on the real number line (the continuum). They include irrational and rational numbers. Denoted $\mathbb{R}$.

POLYNOMIALS

- A polynomial in one variable is an expression of the form
$$a_n x^n + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$
where $a_0, a_1, a_2, \ldots, a_n$ are real numbers and n is a natural number called the **degree** of the polynomial.
- A polynomial in multiple variables is a sum of terms in which each variable is raised to a natural number exponent.
- A **linear polynomial** is one in which all variables have an exponent of 1 and appear in separate terms.
- A polynomial is factored when it is written as a product of polynomials (generally of lower degree) that cannot be further factored.

CALCULUS

- Power rule: $\frac{d}{dx}[x^n] = nx^{n-1}$
- Product rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- Quotient rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
- Chain rule: $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$
- The derivative of a polynomial function $f(x) = a_n x^n + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ can be found using the power rule:
$$f'(x) = an_n x^{n-1} + \cdots + 3a_3 x^2 + 2a_2 x + a_1$$

TRIGONOMETRY

- Useful identities:
$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$
$$\sin^2(x) + \cos^2(x) = 1 \qquad \tan^2(x) - \sec^2(x) = -1 \qquad \cot^2(x) - \csc^2(x) = -1$$
- A summary of trigonometric functions and their properties:

Function	Domain	Range	$f(0)$	$f(\frac{\pi}{4})$	$f(\frac{\pi}{3})$	$f(\frac{\pi}{2})$	$f'(x)$
$\sin(x)$	$\mathbb{R}$	$[-1, 1]$	0	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\cos(x)$
$\cos(x)$	$\mathbb{R}$	$[-1, 1]$	1	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\sin(x)$
$\tan(x)$	$\mathbb{R} - \{\frac{n\pi}{2} : n$ is an odd integer$\}$	$\mathbb{R}$	0	1	$\sqrt{3}$	undef	$\sec^2(x)$
$\csc(x)$	$\mathbb{R} - \{n\pi : n$ is an any integer$\}$	$(-\infty, -1] \cup [1, +\infty)$	undef	$\sqrt{2}$	$\frac{2\sqrt{3}}{2}$	1	$-\csc(x)\cot(x)$
$\sec(x)$	$\mathbb{R} - \{\frac{n\pi}{2} : n$ is an odd integer$\}$	$(-\infty, -1] \cup [1, +\infty)$	0	$\sqrt{2}$	2	undef	$\sec(x)\tan(x)$
$\cot(x)$	$\mathbb{R} - \{n\pi : n$ is an any integer$\}$	$\mathbb{R}$	undef	1	$\frac{\sqrt{3}}{3}$	0	$-\csc^2(x)$

SYSTEMS OF LINEAR EQUATIONS

- A **linear equation** in n variables $x_1, x_2, \ldots, x_n$ is one of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

  where $a_1, a_2, \ldots, a_n$ are numbers called **coefficients** and b is a number.

- A **solution** to a linear equation is a set of values for the variables that make the equation true. A linear equation has multiple solutions.

  $x = 1, y = 2$ is a solution to $5x - 3y = -1$ since $5(1) - 3(2) = -1$

  $x = 1, y = 2$ is NOT a solution to $3x + 4y = 2$ since $3(1) + 4(2) = 11 \neq 2$

- A **system of linear equations** is a collection of linear equations in the same variable(s). A **solution** to a system of linear equations is a set of values for the variables that make all equations in the system true.

$$\begin{array}{rcrcr} 4x & - & 2y & = & 14 \\ -x & + & y & = & -5 \end{array} \text{ has solution } x = 2, y = -3 \text{ since } \begin{array}{rcrcr} 4(2) & - & 2(-3) & = & 14 \\ -2 & + & (-3) & = & -5 \end{array}$$

- A system of linear equations may have 0 solutions, one solution, or infinitely many solutions. No other possibilities exist; that is, a system cannot have exactly two solutions.

- **Substitution method** for solving systems of equations (two equation, two variable version):
  1. Choose either variable in either equation. Solve for this variable in terms of the other variables.
  2. Substitute this expression for the variable into the other equation to create an equation in only one variable. Solve this equation.
  3. Use the solution for the first variable to solve for the second. Evaluate both equations with your solution to check.

System to solve	$\begin{array}{rcrcr} 4x & - & 2y & = & 14 \\ -x & + & y & = & -5 \end{array}$
Solving for y in terms of x in second equation	$y = -5 + x$
Substitute expression for y into first equation	$4x - 2(-5 + x) = 14$
Solve for x	$4x + 10 - 2x = 14$
	$2x = 4$
	$x = 2$
Solve for y	$y = -5 + x = -5 + 2 = -3$

- **Elimination method** for solving systems of equations (two equation, two variables):
  1. Multiply entire equations as needed so the coefficients of either variable are equal.
  2. Subtract one equation from the other to obtain an equation in one variable. Solve for this variable.
  3. Use the solution for the first variable to solve for the second. Evaluate both equations with your solution to check.

System to solve	$\begin{array}{rcrcr} 4x & - & 2y & = & 14 \\ -x & + & y & = & -5 \end{array}$
Multiply second equation by -2 so coefficients of y are equal	$\begin{array}{rcrcr} 4x & - & 2y & = & 14 \\ 2x & - & 2y & = & 10 \end{array}$
Subtract second equation from first and solve for x	$2x + 0y = 4$
	$x = 2$
Use either original equation to solve for y	$-x + y = -5$
	$-2 + y = -5$
	$y = -3$

SUGGESTED REVIEW PROBLEMS

Please attempt to solve all problems on your own before checking the solutions on the next page!

1. Which of the following is a solution of the linear equation $4x_1 + 2x_2 - 5x_3 + x_4 = 4$? More than one answer is possible.

   (a) $x_1 = 0, x_2 = 6, x_3 = 0, x_4 = -2$

   (b) $x_1 = 0, x_2 = -8, x_3 = 6, x_4 = 7$

   (c) $x_1 = 2, x_2 = 3, x_3 = 1, x_4 = -5$

   (d) $x_1 = -1, x_2 = 6, x_3 = 1, x_4 = 1$

2. Solve the following systems of equations. Use any method you prefer.

   (a) $\begin{array}{rcrcr} 4x & - & 3y & = & 11 \\ -6x & + & y & = & -6 \end{array}$

   (b) $\begin{array}{rcrcr} 9x & + & y & = & 27 \\ -2x & - & 5y & = & -6 \end{array}$

   (c) $\begin{array}{rcrcr} 2x & + & 6y & = & 10 \\ -4x & - & 12y & = & -20 \end{array}$

   (d) $\begin{array}{rcrcr} 3x & + & 8y & = & 2 \\ & & 3y & = & 12 \end{array}$

   (e) $\begin{array}{rcrcr} x & - & y & = & 8 \\ 2x & - & 2y & = & 12 \end{array}$

   (f) $\begin{array}{rcrcr} 12x & - & 6y & = & -12 \\ & & 5x & = & 5y \end{array}$

3. Which of the following functions, if any, is the inverse of $f(x) = 3x + 8$?

   (a) $f^{-1}(x) = 3x - 8$

   (b) $f^{-1}(x) = \dfrac{1}{3x + 8}$

   (c) $f^{-1}(x) = \dfrac{x - 8}{3}$

   (d) $f^{-1}(x) = -3x - 8$

4. Simplify the following expressions completely:

   (a) $6(x - 2)^2 + 3[x - (8 - x)] + 8x^2 - 10$

   (b) $\dfrac{(a^3b^{-4}c^7)^2(a^5b^8c^{-1/2})^0}{(ab^3)^{-4}c^6}$

5. Factor the polynomials:

   (a) $x^2 - 3x - 4$

   (b) $x^3 + 2x^2 - 3x - 6$

6. Solve the polynomial equations and list the multiplicity of each solution:

   (a) $3x^2 + 14x = 5$

   (b) $8x^3 - 32x = 0$

7. Sketch the graph of the linear functions:

   (a) $y = 3x - 4$

   (b) $6x + 3y = 3$

8. Sketch a graph of both equations in problem 2a on the same grid. What do you notice about the intersection of the lines, when compared to your solution to this problem? Repeat for problem 2e.

9. Without solving for x, find $\cos(x)$ if $\sin(x) = \frac{1}{2}$ and x is between 0 and $\frac{\pi}{2}$.

10. Find the derivative of the functions:

    (a) $f(x) = 6x^3 - 7x^2 + x - 9$

    (b) $f(x) = \sin^2(x)\cos(x)$

    (c) $f(x) = \dfrac{\sqrt{x + 1}}{3x^2}$

    (d) $f(x) = (4x^3 + 5x^2 - 11)^{30}$

SUGGESTED REVIEW PROBLEM SOLUTIONS

1. (c) and (d) are both solutions

2. (a) $x = \frac{1}{2}, y = -3$
   (b) $x = 3, y = 0$
   (c) infinitely many solutions
   (d) $x = -10, y = 4$
   (e) no solution
   (f) $x = -2, y = -2$

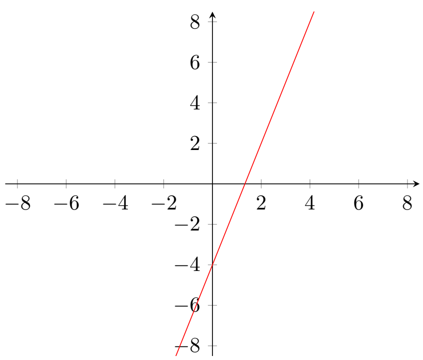
3. (c) $\frac{x-8}{3}$

4. (a) $14x^2 - 18x - 10$
   (b) $a^{10}b^4c^8$

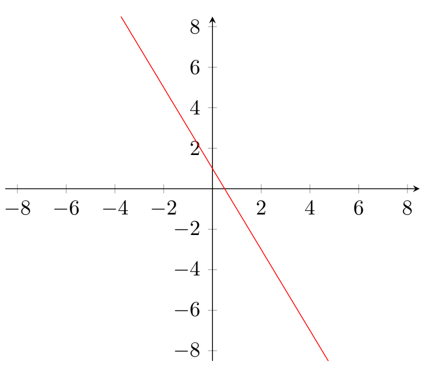
5. (a) $(x-4)(x+1)$
   (b) $(x^2-3)(x+2)$

6. (a) $x = -5, x = \frac{1}{3}$ both with multiplicity 1
   (b) $x = 4$ with multiplicity 1, $x = 0$ with multiplicity 2

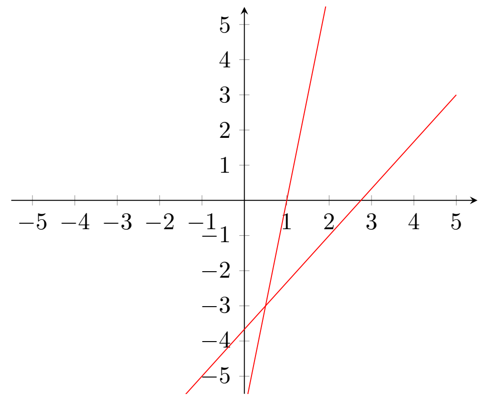
7. (a)



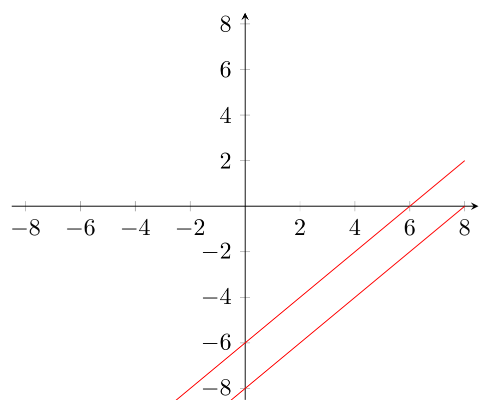
(b)



8. The lines in problem 2a intersect at the point $(\frac{1}{2}, -3)$, the solution of the system.



The lines in problem 2e are parallel, and do not intersect. There is no solution to this problem, because there is no point (x, y) on both lines.



9. Using $\sin^2(x) + \cos^2(x) = 1$, we find $\cos(x) = \frac{\sqrt{3}}{2}$

10. (a) $f'(x) = 18x^2 - 14x + 1$
    (b) $f'(x) = 2\sin(x)\cos^2(x) - \sin^3(x)$
    (c) $f(x) = \dfrac{-3x-4}{6x^3\sqrt{x+1}}$
    (d) $f(x) = 30(4x^3 + 5x^2 - 11)^{29}(12x^2 + 10x)$