Math 170 Calculus I

Final Exam Review ∘ Final Exam: May 9, 2018 8 - 10 am

The final exam is cumulative, and will include all sections covered in class in Chapters 1 through 5.

You will be expected to know:

- how to evaluate a limit of the form $\lim_{x\to a} f(x)$ where $f(x)$ is defined at a
- how to determine if a limit of the form $\lim_{x\to a} f(x)$ where $f(x)$ is not defined at a exists, and how to evaluate it using one sided limits and/or algebraic simplification
- how to calculate limits at infinity
- the definition of continuity at a point and continuity on an interval
- how to find discontinuities of a function, and identity their type
- how to find vertical and horizontal asymptotes of a function
- that $f'(a)$ indicates the slope of the tangent line of $f(x)$ at $x = a$
- how to find the derivative of a function using the limit definition
- how to find the derivative of a function using the power rule, chain rule, product rule, and/or quotient rule, as appropriate
- how to find values of x where $f'(x)$ does not exist
- the derivatives of the six trigonometric functions and the six inverse trigonometric functions
- the derivatives of exponential and logarithmic functions
- how to find y' and y'' for an implicitly defined function y
- how to use the Intermediate Value Theorem to show that a continuous function has a root within a given interval
- how to set up and solve related rates problems
- how to expand a logarithmic expression into a sum and difference of simpler functions
- how to apply logarithmic differentiation
- how to evaluate limits of indeterminate forms using l'Hopital's rule
- the definition of critical value and inflection point, and how to calculate them
- how to find the intervals where a function is increasing and where it is decreasing
- how to find the intervals where a function is concave up and where it is concave down
- how to find the relative extrema and absolute extrema of a function, both in general and on a specified interval
- how to use information about intercepts, asymptotes, concavity, and critical values to sketch a graph of a polynomial or rational function
- how to set up and solve optimization (applied max/min) problems

- the hypotheses and conclusions of Rolle's and Mean Value Theorems, and how to find the values predicated by the theorems
- the geometric interpretation of the definite integral
- how to evaluate an indefinite integral using simple derivative formulas and/or substitution, and how to check that your answer is correct
- how to evaluate a definite integral using simple derivative formulas and/or substitution
- how to evaluate an integral involving an absolute value expression
- how to use the Fundamental Theorem of Calculus, Part II to find the derivative of a definite integral with respect to t and an upper bound of x

Solutions to the review problems will be posted on the course website the day before the final exam (Tuesday). Please make a serious attempt to solve all problems on your own before looking at the solutions.

Review Problems

It is highly recommended that you work on as many of these problems as possible. This is not an exhaustive list of the types of questions you might see. You should also review in class exams, review sheets, and homework problems.

1. Find the following limits:

   (a) $\lim_{x\to -1} \dfrac{x^3 - x^2}{x-1}$

   (b) $\lim_{x\to 2^-} \dfrac{x+2}{x-2}$

   (c) $\lim_{x\to\infty} \dfrac{2x^2-6}{x^2+5x}$

   (d) $\lim_{x\to\infty} \dfrac{x^3-x^2+10}{3x^2-4x}$

   (e) $\lim_{x\to 0} \dfrac{x\sin(x)}{1-\cos(x)}$

   (f) $\lim_{x\to 0} \ln(\sin(2x)) - \ln(\tan(x))$

   (g) $\lim_{x\to 0^+} \dfrac{\cot(x)}{\ln(x)}$

   (h) $\lim_{x\to\infty} x\sin(\pi/x)$

2. Find any values of x (if they exist) where the function $f(x)$ is not continuous:

   (a) $f(x) = \dfrac{4x+1}{x^2-1}$

   (b) $f(x) = |x^2+3|$

   (c) $f(x) = \dfrac{x+5}{|x^2+5x|}$

   (d) $f(x) = e^{\ln(x)}$

3. Find the average value of the function $f(x) = \dfrac{1}{1+x^2}$ on the interval $[1, \sqrt{3}]$.

4. Use the limit definition of the derivative to find the slope of the tangent line to the graph of $f(x) = x^2+1$ at a general x value. Then, use it to find the slope of the tangent line to the graph of f at $x = 4$. Finally, find a value of x where the function is perpendicular to the line $y = \frac{1}{3}x - 4$.

5. Use the limit definition of the derivative to find $f'(x)$ when $f(x) = \sqrt{x-3}$

6. Find $f'(x)$ for each of the following:

   (a) $f(x) = \sin^3(x) + 4\cos(x)$

   (b) $f(x) = \sqrt{5x-2}(x+3)^2$

   (c) $f(x) = \left(\dfrac{4x-2}{x^3}\right)^2$

   (d) $f(x) = \dfrac{4x+7x^2}{2\pi}$

   (e) $f(x) = \tan^3(x^4)$

   (f) $f(x) = \sqrt[3]{5+\sqrt{x}}$

   (g) $f(x) = 2xe^{\sqrt{x}}$

   (h) $f(x) = x + \csc(x^2+1)$

   (i) $f(x) = \cos^3\left(\dfrac{x}{x+1}\right)$

   (j) $f(x) = \dfrac{6}{1+3e^x}$

   (k) $f(x) = \cos^{-1} x^2$

   (l) $f(x) = \sqrt[3]{\ln(x)+1}$

7. Find y' when

   (a) $xy = x - y$

   (b) $x^3 + xy + y^3 = x$

   (c) $3xy^2 - 6x + 3y^3 = 9$

   (d) $\frac{1}{y} + \frac{1}{x} = 1$

8. Find the derivative of $y = \dfrac{x^3}{\sqrt{x^2+3}}$ using logarithmic differentiation.

9. Two parallel sides of a rectangle are being lengthened at a rate of 2 in/sec, while the other two sides are shortened in such a way that the figure remains a rectangle with area 50 in^2. What is the rate of change of the perimeter when the length of an increasing side is 5 in? Is the perimeter increasing or decreasing?

10. For each of the following functions, find (a) the intervals on which the function is increasing or decreasing and (b) the intervals on which the function is concave up or concave down. (Hint: Your answers may not be "nice" - do not expect integer values for the endpoints of your interval in either problem)

(a) $f(x) = \dfrac{x-2}{(x^2-x+1)^2}$

(b) $f(x) = x^3 \ln(x)$

11. Sketch a graph of the following functions by calculating critical values, inflection points, intercepts, intervals of increasing and decreasing, intervals of concavity, asymptotes, etc.

(a) $f(x) = x^2 - 3x - 4$

(b) $f(x) = \dfrac{x-3}{x^2-9}$

(c) $f(x) = \dfrac{2x}{x^2+4}$

(d) $f(x) = x(x^2-1)$

12. A rectangular area of 3200 square feet is to be fenced off. Two opposite sides will use fencing costing \$1 per foot, and the remaining sides will use fencing costing \$2 per foot. Find the dimensions of the rectangle with the lowest possible cost.

13. A closed box with a square base is to be constructed using 10 m^2 of cardboard. Assuming all material is used, determine the maximum volume the box can have.

14. Verify that the hypotheses of Rolle's Theorem are satisfied on the interval $[-1, 3]$ for the function $f(x) = \ln(4 + 2x - x^2)$, and find all values of c in that interval that satisfy the conclusion of the theorem.

15. Find the integrals:

(a) $\int (4x^3 - 6x + 8)\, dx$

(b) $\int_1^4 \dfrac{4 - 3x + 6x^2}{x^2}\, dx$

(c) $\int_1^{\sqrt{2}} xe^{-x^2}\, dx$

(d) $\int_{-1}^1 \dfrac{2}{1+x^2}\, dx$

(e) $\int \tan(2\theta)\, d\theta$

(f) $\int_{1/2}^1 \dfrac{1}{2x}\, dx$

(g) $\int_{-2}^{-1} \dfrac{x}{(x^2+2)^3}\, dx$

(h) $\int_0^2 |2x-3|\, dx$

(i) $\int x(1+x^3)\, dx$

16. Find $F'(x)$ when

$$F(x) = \int_1^x \frac{\sin(t^3-1)}{\sqrt[3]{27t^6-1}\,\cos(t^3-1)}\, dt$$

Quick Reference on Covered Material

Limits

- One sided limits. The limit of a function $f(x)$ as x approaches a from the right or from the left are denoted by

$$\lim_{x\to a^+} f(x) \qquad \text{and} \qquad \lim_{x\to a^-} f(x)$$

  respectively.

- If both one sided limits exist, and are both equal to the same finite number L, then we say the limit of $f(x)$ as x approaches a exists, and is equal to L:

$$\lim_{x\to a} f(x) = L$$

  Intuitively, this means that the value of the function gets arbitrarily close to L when x is close to a.

- If either of the one sided limits does not exist, or if the one sided limits are not equal, then the limit does not exist.

- The expressions

$$\lim_{x\to a} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to a} f(x) = \infty$$

  mean that the value of the function $f(x)$ decreases or increases, respectively, without bound as x gets close to a.

- The expressions

$$\lim_{x\to \infty} f(x) = L \qquad \text{and} \qquad \lim_{x\to -\infty} f(x) = L$$

  mean that the value of the function $f(x)$ gets close to L as x increases or decreases, respectively.

- Properties of limits: Assuming that the limits of $f(x)$ and $g(x)$ as $x \to a$ exist, then

  1. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
  2. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
  3. $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f(x)/ \lim_{x\to a} g(x) \qquad$ (when $\lim_{x\to a} g(x) \neq 0$)
  4. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

- The formal definition of a limit: Let $f(x)$ be defined on an interval near $x = a$, except possibly at $x = a$. Then $\lim_{x\to a} f(x) = L$ if, for any $\varepsilon > 0$, we can find some $\delta > 0$ so that $|x - a| < \delta$ implies $|f(x) - L| < \varepsilon$. (You do not need to know this for the final exam)

- $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$

Continuity

- A function $f(x)$ is said to be continuous at $x = a$ if all three of the following conditions hold:
  1. $f(a)$ is defined
  2. $\lim_{x \to a} f(x)$ exists
  3. $\lim_{x \to a} f(x) = f(a)$
- A function is said to be continuous on an interval if it is continuous at every point in that interval.
- If any of the three properties in the definition of continuity at a point fail, the function is said to be discontinuous at that point.
- There are three types of discontinuities. A removable discontinuity results when the function is not defined at a point $x = a$, but its limit exists there. A jump discontinuity results when the one sided limits at a point do not agree (the function may or may not be defined at this point). An infinite discontinuity results when the one sided limits are infinite (either positive or negative, or possibly one of each).
- Polynomials are always continuous everywhere.
- Rational functions are continuous on their domains, but have discontinuities where their denominator is equal to 0. These could be any of the three types of discontinuities - we'd need to check the one sided limits to decide.
- The composition $f(g(x))$ of two continuous functions is continuous.
- The functions $\sin(x)$ and $\cos(x)$ are continuous everywhere. $\tan(x)$ is continuous at all x where $\cos(x) \neq 0$ (it has infinite discontinuities where $\cos(x) = 0$).
- The Intermediate Value Theorem says that if $f(x)$ is continuous on a closed interval $[a, b]$, and k is a number between $f(a)$ and $f(b)$, then there must be some value c between a and b so that $f(c) = k$.

Derivatives

- The derivative of a function $f(x)$ at a point $x = a$ is the slope of the tangent line to $f(x)$ at $x = a$.
- The limit definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  A function is called differentiable when this limit exists. If the limit does not exist for a particular value of x, the function is not differentiable there.

- Properties of derivatives: Assuming that $f(x)$ and $g(x)$ are both differentiable, then
  1. $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) + g'(x)$
  2. $\frac{d}{dx}[cf(x)] = cf'(x)$ for a constant c.
- Differentiable functions are continuous. Continuous functions may not be differentiable - $|x|$ is an example, because it is continuous everywhere, but not differentiable at 0.
- Derivative rules: Assuming $f(x)$ and $g(x)$ are differentiable, then
  1. The power rule: $\frac{d}{dx}[x^n] = nx^{n-1}$ for any n
  2. The product rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
  3. The quotient rule: $\frac{d}{dx}[\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
  4. The chain rule: $\frac{d}{dx}[f(g(x))] = g'(x)f'(g(x))$
- The derivative y' of functions that are not easily written in the form $y = f(x)$ may be found implicitly. The process involves taking derivatives with respect to x, using differentiation rules as needed. The key is to remember that y is a function of x, so when taking the derivative of y we need to use the chain rule.
- The derivative of $\ln(x)$ is $1/x$. When finding the derivative of $\ln(f(x))$, we use the chain rule:

$$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$$

- The derivative of e^x is e^x. When finding the derivative of $e^{f(x)}$, we use the chain rule:

$$\frac{d}{dx}[e^{f(x)}] = f'(x)e^{f(x)}$$

- Derivatives of trigonometric functions:

$$\frac{d}{dx}[\sin(x)] = \cos(x) \qquad \frac{d}{dx}[\cos(x)] = -\sin(x) \qquad \frac{d}{dx}[\tan(x)] = \sec^2(x)$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x) \qquad \frac{d}{dx}[\cot(x)] = -\csc^2(x) \qquad \frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

- Derivatives of inverse trigonometric functions:

$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\cos^{-1}(x)] = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}[\cot^{-1}(x)] = -\frac{1}{1+x^2} \qquad \frac{d}{dx}[\csc^{-1}(x)] = -\frac{1}{|x|\sqrt{x^2-1}}$$

Applications of Derivatives

- Related rate problems involve finding the rate at which one quantity is changing, when this quantity is related to other quantities whose rates of change are known. The basic strategy is:
  1. Draw a picture, when possible, and label (make up variables for) all quantities that vary with time, as well as any other quantities that seem relevant.
  2. Find an equation relating the quantities.
  3. Differentiate both sides of this equation, then solve for the unknown rate by substituting the known values into the equation.

- We can approximate the value of non-linear functions (functions whose graph is not a straight line) by local linear approximation. If we know the value of the function at a point x_0, we can approximate the value of the function at x using the formula

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

- We may be able to evaluate difficult limits by L'Hopital's rule:
  1. To evaluate $\lim_{x\to a} \frac{f(x)}{g(x)}$ when both $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$, or both $f(x) \to 0$ and $g(x) \to 0$, as $x \to a$:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

  2. To evaluate the limit of an expression $f(x)g(x)$ where one function approaches $\pm\infty$ and the other approaches 0, write the product as a ratio, then apply L'Hopital's rule:

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{1/g(x)} = \lim_{x\to a} \frac{g(x)}{1/f(x)}$$

- The position, velocity, and acceleration of a particle are linked by the derivative. If $s(t)$ is the position of a particle at time t, then $s'(t)$ is its velocity, and $s''(t)$ is its acceleration.

- The critical values of a function $f(x)$ are values of x where $f'(x) = 0$ or $f'(x)$ is undefined. If $x = a$ is a critical value of f, then $(a, f(a))$ is a critical point.

- The inflection points of a function $f(x)$ occur at values of x where $f''(x) = 0$ or $f''(x)$ is undefined. If $x = a$ is such a value, then $(a, f(a))$ is an inflection point.

- Maximum and minimum values of a function defined on an open interval (including the entire real number line) occur at critical points. A function defined on an open interval may not have a minimum or maximum (it will typically have at least one, unless it's a horizontal line).

- A function defined on a closed interval will have both a maximum and minimum value on that interval (this is the Extreme Value Theorem). These occur at any critical values within the interval, or at the endpoints of the interval.

- Graphing functions (of any type) requires a series of steps to be taken:

  1. Find x and y intercepts. The x intercepts are found by setting the function equal to 0 and solving for x; the y intercept (there's only one!) is found by evaluating the function at 0.
  2. Find the critical values, then the critical points.
  3. Find the intervals where the function is increasing or decreasing. Choose a value between consecutive critical values a and b, and plug into the first derivative. If the result is positive, the function is increasing on the interval (a, b). If the result is negative, the function is decreasing.
  4. Find the inflection points.
  5. Find the intervals where the function is concave up or concave down. Choose a value between consecutive inflection points a and b, and plug into the second derivative. If the result is positive, the function is concave up on the interval (a, b). If the result is negative, the function is concave down.
  6. For rational functions: Check for vertical asymptotes by finding values of x where the function is undefined (set the denominator equal to 0, and solve). Take the one-sided limits of the function as x approaches each of these values - if they're $\pm\infty$, the function has a vertical asymptote. Polynomials do not have vertical asymptotes.
  7. Check for horizontal asymptotes by finding the limits of the function as x approaches $\pm\infty$.
  8. Use all of the above information to sketch the graph of the function. Plot any intercepts, critical points, and inflection points (as long as the function is defined there), then fill in the rest of the graph, taking concavity and direction into account.

- Applied maximum and minimum problems involve finding a maximum or minimum possible value when certain quantities are restricted. The basic strategy is:

  1. Draw a picture, when possible, and label (make up variables for) all quantities relevant to the problem.
  2. Find an equation for the quantity that needs to be maximized or minimized. These will typically involve too many variables to solve directly.
  3. Find a "restriction equation", that uses the stated conditions to relate known and unknown quantities.
  4. Use the restriction equation to write one variable in terms of the other, and make a substitution in the original equation to be maximized or minimized.
  5. Take the derivative of both sides of the equation to be maximized or minimized, and find any critical values.
  6. Check each critical value for a minimum or maximum, as needed. Ignore any critical values that don't make sense for the original problem. If the problem can be interpreted as occurring over a closed interval, check the endpoints of that interval as well.
  7. Be sure to find what the problem was actually asking for - you may need to go back find additional values.

- Rolle's Theorem states that if $f(x)$ satisfies the following properties:

  1. continuous on the closed interval $[a, b]$
  2. differentiable on the open interval (a, b)
  3. $f(a) = f(b) = 0$

  then there exists some number c between a and b so that $f'(c) = 0$.

- The Mean Value Theorem states that if $f(x)$ satisfies the following properties:

  1. continuous on the closed interval $[a, b]$
  2. differentiable on the open interval (a, b)

  then there exists some number c between a and b so that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

- An implication of the Mean Value Theorem is the Constant Difference Theorem: If two functions have the same derivative, then they differ only by a constant value.

Antiderivatives and Integration

- A function $F(x)$ is called an antiderivative of $f(x)$ if $F'(x) = f(x)$ for all x
- The indefinite integral of $f(x)$ is $\int f(x)dx = F(x) + C$, where $F(x)$ is an antiderivative of $f(x)$
- Properties of the integral:
  1. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ for a constant c
  2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int g(x)dx$
  3. $\int_a^a f(x)dx = 0$
  4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
  5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a \leq c \leq b$
  6. If $f(x) \leq g(x)$ for all x in $[a.b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
- The Fundamental Theorem of Calculus states that if f is a continuous function on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then
$$\int_a^b f(x)dx = F(b) - F(a)$$
- Note the major difference between indefinite and definite integrals: the indefinite integral $\int f(x)dx$ is a function (the antiderivative of f) while the definite integral $\int_a^b f(x)dx$ is a number.
- The second part of the FTC says that if
$$F(x) = \int_a^x f(t)\ dt$$
then $F'(x) = f(x)$
- Integrals (indefinite or definite) of the form
$$\int f(g(x))g'(x)dx$$
can be evaluated by substitution. Let $u = g(x)$, so $du = g'(x)dx$, and the integral becomes $\int f(u)du$. After finding the integral in terms of u, replace all instances of u with $g(x)$, and evaluate at the limits for definite integral.
- We can define the natural logarithm as an integral:
$$\ln(x) = \int_1^x \frac{1}{t}dt$$

Some Worked Examples

1. Implicit Differentiation (Section 3.1):
   Find y' when $x^3y^2 - 3x^2y + x = 1$

The key is to remember that y is a function of x, and that we're differentiating with respect to x. In most cases, we won't know what that function is, but we can still take it's derivative, and call it y'. To find a derivative of a term involving powers of y (or even more interesting things), just remember that y is a function of x, and use the chain rule. As an example, suppose $y = x^5$. If we wanted to find the derivative of y^2, we'd use the chain rule:

$$\frac{d}{dx}(y^2) = \frac{d}{dx}((x^5)^2) = 2(x^5)\frac{d}{dx}(x^5) = 2yy'$$

So in general, the derivative of y is simply y', the derivative of y^2 is $2yy'$, etc. If a term involves both x and y, we use the product rule. We'll take the derivative by term:

$$\frac{d}{dx}(x^3y^2) = \frac{d}{dx}(x^3)y^2 + x^3\frac{d}{dx}(y^2) = 3x^2y^2 + x^3 2yy'$$

$$\frac{d}{dx}(-3x^2y) = \frac{d}{dx}(-3x^2)y - 3x^2\frac{d}{dx}(y) = -6xy - 3x^2y'$$

$$\frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}(1) = 0$$

After taking the derivative (with respect to x) of both sides, we're left with

$$3x^2y^2 + x^3 2yy' - 6xy - 3x^2y' + 1 = 0$$

To solve for y', first get all terms involving a y' on one side of the equation, and all other terms on the other side:

$$x^3 2yy' - 3x^2y' = -3x^2y^2 + 6xy - 1$$

Factor out the y', then divide to find the expression (in terms of x and y) for y':

$$y'(x^3 2y - 3x^2) = -3x^2y^2 + 6xy - 1$$

$$y' = \frac{-3x^2y^2 + 6xy - 1}{x^3 2y - 3x^2}$$

2. Related Rates (Section 3.4):

A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 ft/s. How rapidly is the area enclosed by the ripple increasing after 10 s?

We first need to set up some variables, and figure out what it is that we're looking for. Let r be the radius of the circle enclosed by the ripple, and let $A = \pi r^2$ be the area of that circle. We want to find $\frac{dA}{dt}$, the rate at which the area is changing with time, when $t = 10$. Differentiating both sides of the area equation with respect to time gives us

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

We know one rate: the radius is increasing at 3 ft/s, so $\frac{dr}{dt} = 3$ ft/s. At the moment when $t = 10$, the radius must be $r = 3 \cdot 10 = 30$ ft. So, at $t = 10$, we have

$$\frac{dA}{dt} = 2\pi(30)(3) = 180\pi \text{ ft/s}$$

3. L'Hopital's Rule (Section 3.6):

$\lim_{x \to 0^+} \frac{\sin(x)}{x^2}$

This is an indeterminate form of type $0/0$, since both the numerator and denominator approach 0 as $x \to 0^+$. We can apply L'Hopital's rule, meaning that we'll take the derivative of the numerator and denominator individually (no quotient rule!), then take the limit:

$$\lim_{x \to 0^+} \frac{\sin(x)}{x^2} = \lim_{x \to 0^+} \frac{\cos(x)}{2x}$$

This is no longer an indeterminate form, so we can't apply the rule again. But, we can now see that the limit is ∞.

4. L'Hopital's Rule (Section 3.6):

$\lim_{x \to 0^+} \tan(x)\ln(x)$

This is an indeterminate form of type $0 \cdot \infty$, since $\tan(x) \to 0$ while $\ln(x) \to -\infty$ as $x \to 0^+$. In order to apply L'Hopital's rule, we need to rewrite the expression as an indeterminate form of type $0/0$ or ∞/∞, using the fact that $\tan(x) = 1/1/\tan(x) = 1/\cot(x)$:

$$\lim_{x \to 0^+} \tan(x)\ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{\cot(x)}$$

We can now apply the rule, as many times as needed, until we can take the limit:

$$\lim_{x \to 0^+} \frac{\ln(x)}{\cot(x)} = \lim_{x \to 0^+} \frac{1/x}{-\csc^2(x)} = \lim_{x \to 0^+} \frac{-\sin^2(x)}{x} = \lim_{x \to 0^+} \frac{-2\sin(x)\cos(x)}{1} = 0$$

5. Graphing Functions (Chapter 4):

Graph the function $f(x) = \frac{x^2}{x^2-4}$

- x intercepts:
  $f(x) = 0$ when $x = 0$ (just need to look at the numerator) so the only x intercept is $(0, 0)$
- y intercepts:
  $f(0) = 0$ so the y intercept is $(0, 0)$
- Critical values:

$$f'(x) = \frac{(x^2-4)(2x) - (x^2)(2x)}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2}$$

  $f'(x) = 0$ where its numerator is 0, so at $x = 0$
  $f'(x)$ is undefined where its denominator is 0, so at $x = -2$ and $x = 2$
  The critical values are $-2, 0, 2$
- Critical points:
  $f(-2)$ and $f(2)$ are undefined, so even though there's critical values there, we have no critical points. We do have a critical point at $(0, 0)$, since $f(0) = 0$.
- Increasing and decreasing intervals:
  We have to test $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$. To make things a little easier, notice that the denominator must always be positive. This means that we just need to check the numerator:

$$f'(-10) = \frac{-8(-10)}{\text{a positive number}} > 0 \qquad f'(-1) = \frac{-8(-1)}{\text{a positive number}} > 0$$

$$f'(1) = \frac{-8(1)}{\text{a positive number}} < 0 \qquad f'(10) = \frac{-8(10)}{\text{a positive number}} < 0$$

  So the function is increasing on the intervals $(-\infty, -2)$ and $(-2, 0)$, but decreasing on $(0, 2)$, and $(2, \infty)$.
- Inflection points:

$$f''(x) = \frac{((x^2-4)^2)(-8) - (-8x)(2(x^2-4)(2x))}{(x^2-4)^4} = \frac{24x^2+32}{(x^2-4)^3}$$

  $f''(x) = 0$ where its numerator is 0, which can't happen in this case
  $f''(x)$ is undefined when its denominator is 0, so at $x = -2$ and $x = 2$.
- Concavity:
  We have to test $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$. We again can pick any value from each interval, and test whether or not $f''(x)$ is positive or negative there:

$$f''(-10) > 0 \qquad f''(0) < 0 \qquad f''(10) > 0$$

  So the function is concave up on the intervals $(-\infty, -2)$ and $(2, \infty)$, and concave down on $(-2, 2)$.

- Vertical asymptotes:
  If the function has any vertical asymptotes, they'll occur where $f(x)$ is undefined. This happens when $x = -2$ or $x = 2$, but we still need to check that there is an asymptote there:

$$\lim_{x \to -2^-} \frac{x^2}{x^2 - 4} = \infty \qquad \lim_{x \to -2^+} \frac{x^2}{x^2 - 4} = -\infty$$

$$\lim_{x \to 2^-} \frac{x^2}{x^2 - 4} = -\infty \qquad \lim_{x \to 2^+} \frac{x^2}{x^2 - 4} = \infty$$

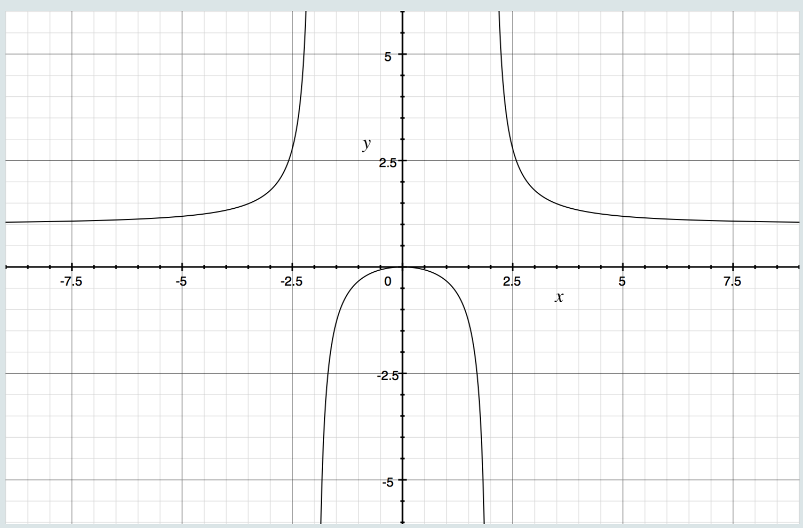
  So we really do have vertical asymptotes $x = -2$ and $x = 2$

- Horizontal asymptotes:

$$\lim_{x \to -\infty} \frac{x^2}{x^2 - 4} = 1 \qquad \lim_{x \to \infty} \frac{x^2}{x^2 - 4} = 1$$

  So we have a horizontal asymptote of $y = 1$

- Graph:



6. Applied Maximum and Minimum Values (Chapter 4.5):

A box with a square base is taller than it is wide. In order to send the box through the US mail, the height of the box and the perimeter of the base can sum to no more than 108 inches. What is the maximum volume of such a box?

Let b be the length of one side of the base of the box, and let h be the height. The volume of the box is given by

$$V = b^2 h$$

which is what we'd like to maximize. Before we can do that, we need a restriction equation. The perimeter of the base is $4b$, and the requirement given tells us

$$4b + h \leq 108$$

Since we're trying to maximize the volume, we should use the largest dimensions possible, and just assume $4b + h = 108$. Then we can write $h = 108 - 4b$, and so

$$V = b^2(108 - 4b) = 108b^2 - 4b^3$$

To find the maximum V, we have to take the derivative:

$$V' = 216b - 12b^2 = 12b(18 - b)$$

So we have critical values at 0 and 18. Taking $b = 0$ doesn't make any sense here, since that means that our box is one dimensional. So the only possibility is $b = 18$, which means $V = 11,664$ cubic inches is the maximum. We could use the first derivative test to make sure this really is a maximum (I'll leave that as an exercise!).

7. The Mean Value Theorem (Chapter 4.8):

Let $f(x) = x^3 + x - 4$. Verify that the hypotheses of the Mean Value Theorem are satisfied on the interval $[-1, 2]$, and find all values of c in that interval that satisfy the conclusion of the theorem.

The function must be continuous on $[-1, 2]$ and differentiable on $(-1, 2)$. Since f is a polynomial, we know that both of those requirements are met.

The theorem states that there is at least one value of c in $(-1, 2)$ where

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{[2^3 + 2 - 4] - [(-1)^3 + (-1) - 4]}{3} = \frac{12}{3} = 4$$

Then c must satisfy

$$4 = f'(c) = 3c^2 + 1$$

There are two possible values of c here, $c = \pm 1$, but only one is in the open interval $(-1, 2)$. So the value of c that satisfies the theorem is $c = 1$.