Math 170 Calculus I

Final Exam Review Solutions

1. Find the following limits:

   (a) $\lim_{x\to -1} \frac{x^3 - x^2}{x-1} = 1$

   (b) $\lim_{x\to 2^-} \frac{x+2}{x-2} = -\infty$

   (c) $\lim_{x\to\infty} \frac{2x^2-6}{x^2+5x} = 2$

   (d) $\lim_{x\to\infty} \frac{x^3 - x^2 + 10}{3x^2 - 4x} = \infty$

   (e) $\lim_{x\to 0} \frac{x\sin(x)}{1-\cos(x)} = 2$

   (f) $\lim_{x\to 0} \ln(\sin(2x)) - \ln(\tan(x)) = \ln(2)$

   (g) $\lim_{x\to 0^+} \frac{\cot(x)}{\ln(x)} = -\infty$

   (h) $\lim_{x\to\infty} x\sin(\pi/x) = \pi$

2. Find any values of x (if they exist) where the function $f(x)$ is not continuous:

   (a) $f(x) = \frac{4x+1}{x^2-1}$, $x = \pm 1$

   (b) $f(x) = |x^2+3|$, cont everywhere

   (c) $f(x) = \frac{x+5}{|x^2+5x|}$, $x = 0, -5$

   (d) $f(x) = e^{\ln(x)}$, $x > 0$

3. Find the average value of the function $f(x) = \frac{1}{1+x^2}$ on the interval $[1, \sqrt{3}]$.

4. Use the limit definition of the derivative to find the slope of the tangent line to the graph of $f(x) = x^2+1$ at a general x value. Then, use it to find the slope of the tangent line to the graph of f at $x = 4$. Finally, find a value of x where the function is perpendicular to the line $y = \frac{1}{3}x - 4$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2+1-x^2-1}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} 2x+h = 2x$$

$$f'(4) = 2(4) = 8$$

The tangent line to the function (and hence the function itself) is perpendicular to the given line when its slope is -3. We have

$$-3 = f'(x) = 2x$$

when $x = -3/2$.

5. Use the limit definition of the derivative to find $f'(x)$ when $f(x) = \sqrt{x-3}$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h-3}-\sqrt{x-3}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h-3}-\sqrt{x-3}}{h} \cdot \frac{\sqrt{x+h-3}+\sqrt{x-3}}{\sqrt{x+h-3}+\sqrt{x-3}} = \lim_{h\to 0} \frac{(x+h-3)-(x-3)}{h(\sqrt{x+h-3}+\sqrt{x-3})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h-3}+\sqrt{x-3})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h-3}+\sqrt{x-3}} = \frac{1}{\sqrt{x+0-3}+\sqrt{x-3}} = \frac{1}{2\sqrt{x-3}} \end{aligned}$$

6. Find $f'(x)$ for each of the following:

   (a) $f(x) = \sin^3(x) + 4\cos(x) = 3\sin^2(x)\cos(x) - 4\sin(x)$

   (b) $f(x) = \sqrt{5x-2}(x+3)^2 = 2\sqrt{5x-2}(x+3) + \frac{5(x+3)^2}{2\sqrt{5x-2}}$

(c) $f(x)=\left(\dfrac{4x-2}{x^3}\right)^2 = 2\left(\dfrac{4x-2}{x^3}\right)\left(\dfrac{-8x^3+6x^2}{x^6}\right)$

(d) $f(x)=\dfrac{4x+7x^2}{2\pi} = \dfrac{2+7x}{\pi}$

(e) $f(x)=\tan^3(x^4) = 12x^3\tan^2(x^4)\sec^2(x^4)$

(f) $f(x)=\sqrt[3]{5+\sqrt{x}} = \dfrac{1}{6}(5+\sqrt{x})^{-2/3}x^{-1/2}$

(g) $f(x)=2xe^{\sqrt{x}} = (2+\sqrt{x})e^{\sqrt{x}}$

(h) $f(x)=x+\csc(x^2+1) = 1-2x\csc(x^2+1)\cot(x^2+1)$

(i) $f(x)=\cos^3\left(\dfrac{x}{x+1}\right) = -3\left(\frac{1}{(x+1)^2}\right)\cos\left(\dfrac{x}{x+1}\right)^2\sin\left(\dfrac{x}{x+1}\right)$

(j) $f(x)=\dfrac{6}{1+3e^x} = -\dfrac{18e^x}{(3e^x+1)^2}$

(k) $f(x)=\cos^{-1}x^2 = -\dfrac{2x}{\sqrt{1-x^4}}$

(l) $f(x)=\sqrt[3]{\ln(x)+1} = \dfrac{1}{3x}(\ln(x)+1)^{-2/3}$

7. Find y' when

(a) $xy=x-y,\quad y'=\dfrac{1-y}{1+x}$

(b) $x^3+xy+y^3=x,\quad y'=\dfrac{1-y-3x^2}{x+3y^2}$

(c) $3xy^2-6x+3y^3=9,\quad y'=\dfrac{6-3y^2}{9y^2+6xy}$

(d) $\frac{1}{y}+\frac{1}{x}=1,\quad y'=-\dfrac{y^2}{x^2}$

8. Find the derivative of $y=\dfrac{x^3}{\sqrt{x^2+3}}$ using logarithmic differentiation.

$$y'=\left(\frac{3}{x}-\frac{x}{x^2+3}\right)\left(\frac{x^3}{\sqrt{x^2+3}}\right)$$

9. Two parallel sides of a rectangle are being lengthened at a rate of 2 in/sec, while the other two sides are shortened in such a way that the figure remains a rectangle with area 50 in^2. What is the rate of change of the perimeter when the length of an increasing side is 5 in? Is the perimeter increasing or decreasing?

Let l be the length of a side of the rectangle that's increasing, and let w be the other side. We have

$$P=2l+2w$$

where P is the perimeter of the rectangle, and we know that

$$lw=50$$

since the area is fixed, and so $w=50/l$ and we can rewrite the perimeter equation as

$$P=2l+2(50/l)=2l+100l^{-1}$$

We're given $dl/dt=2$. Taking derivatives (with respect to t) of the perimeter equation:

$$\frac{dP}{dt}=2\frac{dl}{dt}-100l^{-2}\frac{dl}{dt}$$

When $l=5$, we have

$$\frac{dP}{dt}=2(2)-100(5)^{-2}(2)=-4 \text{ in/sec}$$

So, the total perimeter is decreasing.

10. For each of the following functions, find (a) the intervals on which the function is increasing or decreasing and (b) the intervals on which the function is concave up or concave down. (Hint: Your answers may not be "nice" - do not expect integer values for the endpoints of your interval in either problem)

(a) $f(x) = \dfrac{x-2}{(x^2-x+1)^2}$
Decreasing on $(-\infty, \frac{3-\sqrt{5}}{2})$, $(\frac{3+\sqrt{5}}{2}, \infty)$
Increasing on $(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$
Concave down on $(-\infty, 0)$, $(\frac{4-\sqrt{6}}{2}, \frac{4+\sqrt{6}}{2})$
Concave up on $(0, \frac{4-\sqrt{6}}{2})$, $(\frac{4+\sqrt{6}}{2}, \infty)$

(b) $f(x) = x^3 \ln(x)$
Decreasing on $(-\infty, \frac{1}{\sqrt[3]{e}})$
Increasing on $(\frac{1}{\sqrt[3]{e}}, \infty)$
Concave down on $(-\infty, \frac{1}{\sqrt[6]{e^5}})$
Concave up on $(\frac{1}{\sqrt[6]{e^5}}, \infty)$

11. Sketch a graph of the following functions by calculating critical values, inflection points, intercepts, intervals of increasing and decreasing, intervals of concavity, asymptotes, etc.
Use a graphing calculator or Wolfram Alpha to check that your graphs are correct.

12. A rectangular area of 3200 square feet is to be fenced off. Two opposite sides will use fencing costing \$1 per foot, and the remaining sides will use fencing costing \$2 per foot. Find the dimensions of the rectangle with the lowest possible cost.
If y is the length of a side that costs \$2 per foot, and x is the length of a side costing \$1 per foot, then the dimensions with the lowest possible cost would be $x = 80$ ft and $y = 40$ ft. The restriction equation would be $3200 = xy$ and the cost equation (that we need to find a minimum of) would be $C = 2x + 4y$.

13. A closed box with a square base is to be constructed using 10 m^2 of cardboard. Assuming all material is used, determine the maximum volume the box can have.

14. Verify that the hypotheses of Rolle's Theorem are satisfied on the interval $[-1, 3]$ for the function $f(x) = \ln(4 + 2x - x^2)$, and find all values of c in that interval that satisfy the conclusion of the theorem.
f is continuous on the interval $[-1, 3]$, and its derivative $f'(x) = \frac{2-2x}{4+2x-x^2}$ is defined on $(-1, 3)$. Also, $f(-1) = f(3) = 0$, so all hypothesis are met.

$$0 = f'(c) = \frac{2-2c}{4+2c-c^2} \qquad \Rightarrow \qquad c = 1$$

15. Find the integrals:

(a) $\int (4x^3 - 6x + 8)\, dx = x^4 - 3x^2 + 8x + C$

(b) $\int_1^4 \dfrac{4-3x+6x^2}{x^2}\, dx = 6x - 4x^{-1} - 3\ln(x)\Big]_1^4 = 15 - 3\ln(4)$

(c) $\int_1^{\sqrt{2}} xe^{-x^2}\, dx = \frac{1}{2}\int_{-2}^{-1} e^u\, du = \frac{1}{2}(e^{-1} - e^{-2})$

(d) $\int_{-1}^{1} \dfrac{2}{1+x^2}\, dx = \pi$

(e) $\int \tan(2\theta)\, d\theta = -\frac{1}{2}\ln|\cos(2\theta)| + C$

(f) $\int_{1/2}^{1} \dfrac{1}{2x}\, dx = \frac{1}{2}\ln(x)\Big]_{1/2}^{1} = -\ln(1/2)$

(g) $\int_{-2}^{-1} \dfrac{x}{(x^2+2)^3}\, dx = -\frac{1}{48}$

(h) $\int_0^2 |2x-3|\, dx = \int_0^{3/2} 3 - 2x\, dx + \int_{3/2}^{2} 2x - 3\, dx = 5/2$

(i) $\int x(1+x^3)\, dx = \frac{1}{2}x^2 + \frac{1}{5}x^5 + C$

16. Find $F'(x)$ when

$$F(x) = \int_1^x \frac{\sin(x^3 - 1)}{\sqrt[3]{27x^6 - 1}\,\cos(x^3 - 1)}\,dt$$

There is a typo in this problem! The integrand should have been entirely in terms of t:

$$F(x) = \int_1^x \frac{\sin(t^3 - 1)}{\sqrt[3]{27t^6 - 1}\,\cos(t^3 - 1)}\,dt$$

in which case

$$F'(x) = \frac{\sin(x^3 - 1)}{\sqrt[3]{27x^6 - 1}\,\cos(x^3 - 1)}$$