Problems #4 - 12: Find dy/dx by implicit differentiation.

#4: $x^3 + y^3 = 3xy^2$

We need to use the product rule on the right hand side:

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy^2]$$
$$3x^2 + 3y^2y' = \frac{d}{dx}[3x]y^2 + 3x\frac{d}{dx}[y^2]$$
$$3x^2 + 3y^2y' = 3y^2 + 3x2yy'$$
$$3x^2 + 3y^2y' = 3y^2 + 6xyy'$$

Now solve for y':

$$3y^2y' - 6xyy' = 3y^2 - 3x^2$$
$$y'(3y^2 - 6xy) = 3y^2 - 3x^2$$
$$y' = \frac{3y^2 - 3x^2}{3y^2 - 6xy} = \frac{y^2 - x^2}{y^2 - 2xy}$$

#6: $x^3y^2 - 5x^2y + x = 1$

Scratch work:

$$\frac{d}{dx}[x^3y^2] = \frac{d}{dx}[x^3]y^2 + x^3\frac{d}{dx}[y^2] = 3x^2y^2 + 2x^3yy'$$
$$\frac{d}{dx}[5x^2y] = \frac{d}{dx}[5x^2]y + 5x^2\frac{d}{dx}[y] = 10xy + 5x^2y'$$

Altogether now:

$$\frac{d}{dx}[x^3y^2 - 5x^2y + x] = \frac{d}{dx}[1]$$
$$3x^2y^2 + 2x^3yy' - 10xy - 5x^2y' + 1 = 0$$
$$2x^3yy' - 5x^2y' = 10xy - 3x^2y^2 - 1$$
$$y'(2x^3y - 5x^2) = 10xy - 3x^2y^2 - 1$$
$$y' = \frac{10xy - 3x^2y^2 - 1}{2x^3y - 5x^2}$$

#8: $x^2 = \frac{x+y}{x-y}$

Avoid quotient rule by multiplying both sides by the denominator and expanding:

$$x^2(x - y) = x + y$$
$$x^3 - x^2y = x + y$$

The x^2y term needs the product rule:

$$\frac{d}{dx}[x^3 - x^2y] = \frac{d}{dx}[x + y]$$
$$3x^2 - 2xy - x^2y' = 1 + y'$$
$$y' + x^2y' = 3x^2 - 2xy - 1$$
$$y' = \frac{3x^2 - 2xy - 1}{1 + x^2}$$

#10: $\cos(xy^2) = y$

Use the chain rule, followed by product rule:

$$\frac{d}{dx}[\cos(xy^2)] = \frac{d}{dx}[y]$$
$$-\sin(xy^2)\frac{d}{dx}[xy^2] = y'$$
$$-\sin(xy^2)(y^2 + 2xyy') = y'$$
$$-\sin(xy^2)y^2 - \sin(xy^2)(2xyy') = y'$$
$$-\sin(xy^2)y^2 = y' + \sin(xy^2)(2xyy')$$
$$-\sin(xy^2)y^2 = y'(1 + 2xy\sin(xy^2))$$
$$y' = -\frac{y^2\sin(xy^2)}{1 + 2xy\sin(xy^2)}$$

Problems #14 - 18: Find d^2y/dx^2 by implicit differentiation.

#14: $x^3 + y^3 = 1$

Start with the first:

$$\frac{d}{dx}[x^3 - y^3] = \frac{d}{dx}[1]$$
$$3x^2 - 3y^2y' = 0$$
$$y' = \frac{3x^2}{3y^2} = \frac{x^2}{y^2}$$

Onto the second, using quotient rule (and simplify):

$$y'' = \frac{(y^2)(2x) - (x^2)(2yy')}{y^4} = \frac{2xy^2 - 2x^2yy'}{y^4} = \frac{2xy - 2x^2y'}{y^3}$$

Now replace y' with an equivalent expression in x, y only and simplify again:

$$y'' = \frac{2xy - 2x^2\frac{x^2}{y^2}}{y^3} = \frac{2xy^3 - 2x^4}{y^5}$$

#16: $xy + y^2 = 2$

Start with the first:

$$\begin{aligned} \frac{d}{dx}[xy + y^2] &= \frac{d}{dx}[2] \\ y + xy' + 2yy' &= 0 \\ y' &= -\frac{y}{x + 2y} \end{aligned}$$

Using quotient rule to find y'':

$$y'' = -\frac{(x + 2y)(y') - y(1 + 2y')}{(x + 2y)^2} = -\frac{xy' + 2yy' - y - 2yy'}{(x + 2y)^2} = -\frac{xy' - y}{(x + 2y)^2}$$

Finally, replace y' and simplify:

$$\begin{aligned} y'' &= -\frac{x\left(-\frac{y}{x+2y}\right) - y}{(x + 2y)^2} \\ &= -\frac{-xy - y(x + 2y)}{(x + 2y)^3} \\ &= -\frac{-xy - xy - 2y^2}{(x + 2y)^3} = \frac{2xy + 2y^2}{(x + 2y)^3} \end{aligned}$$