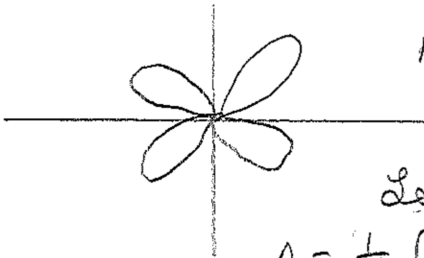
Calculus III Final Exam May 10, 2017
Dr. Platte
Name:____________________

1. (16 points) Find the area of the region in the first quadrant enclosed by a leaf of the four leaf rose r =sin(2θ).



$$\text{Area} = \iint 1\, r\,dr\,d\theta$$

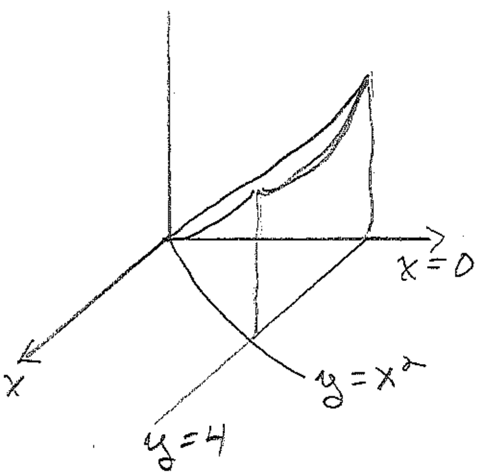
$$A = \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{2}\,d\theta$$

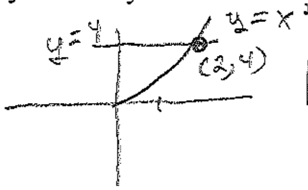
Let $u = 2\theta$, $du = 2d\theta$, $\frac{1}{2}du = d\theta$

$$A = \frac{1}{4}\int \sin^2 u\,du = \frac{1}{4}\left[\frac{1}{2}2\theta - \frac{1}{4}\sin 4\theta\right]\Big|_0^{\frac{\pi}{2}} =$$

$$\frac{1}{4}\cdot\frac{1}{2}\cdot 2\cdot\frac{\pi}{2} = \frac{\pi}{8}$$

2. (16 points) Use a double integral to find the volume of the solid bounded above by $z = x^2$ and below by the region R in the xy plane between $x = 0$, $y = 4$ and $y = x^2$.





$$V = \int_0^2 \int_{x^2}^4 x^2\,dy\,dx$$

$$V = \int_0^2 x^2 y \Big|_{x^2}^4 dx = \int_0^2 (4x^2 - x^4)\,dx$$

$$V = \left(\frac{4}{3}x^3 - \frac{x^5}{5}\right)\Big|_0^2 = \frac{32}{3} - \frac{32}{5} = 32\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$V = 32\cdot\frac{2}{15} = \frac{64}{15}$$

3. (16 points) Find an equation for the tangent plane and parametric equation for the normal line to the surface $z = xy + 3y$ at the point (1,-1,-4)

For normal use $\vec{n} = \langle f_x, f_y, -1 \rangle$ at $(1,-1,-4)$

$\therefore \vec{n} = -i + 4j - k$

Tangent plane: $-(x-1) + 4(y+1) - (z+4) = 0$

$-x + 4y - z + 1 = 0$

normal line: $x = 1 - t$, $y = -1 + 4t$, $z = -4 - t$

4. (16 points) Locate all relative maxima, minima and saddle point, if any for $f(x,y) = xy - x^3 - y^2$.

$f_x = y - 3x^2 = 0$, $f_y = x - 2y = 0$, So $x = 2y$

$\therefore y - 3(2y)^2 = 0$ or $y(1 - 12y) = 0$. Either $y = 0$ or $y = 1/12$

Either $x = 0$ or $x = 1/6$. Critical pts $(0,0)$ $(\frac{1}{6}, \frac{1}{12})$

$f_{xx} = -6x$, $f_{yy} = -2$, $f_{xy} = 1$

$D = 12x - 1$

at $(\frac{1}{6}, \frac{1}{12})$, $D = 1$, $f_{xx} = -1$, $\therefore$ rel. min at $(\frac{1}{6}, \frac{1}{12})$

at $(0,0)$, $D = -1 < 0$, $\therefore$ saddle pt at $(0,0)$

5. (16 points) Let $f(x, y) = x\cos(xy)$ Find the directional derivative of f(x, y) at (1, π/2) in direction of the vector **v** = 5**i** + 12**j** .

$\|\vec{v}\| = \sqrt{25+144} = \sqrt{169} = 13$, so $\vec{u} = \frac{5}{13}\hat{i} + \frac{12}{13}\hat{j}$

$\vec{\nabla} f = (\cos xy - xy\sin xy)\hat{i} + (-x^2\sin xy)\hat{j}$

at $(1, \frac{\pi}{2})$ $\vec{\nabla f} = -\frac{\pi}{2}\hat{i} - \hat{j}$

$D_{\vec{u}} f = \vec{\nabla f} \cdot \vec{u} = -\frac{5\pi}{13 \cdot 2} - \frac{12}{13} = -\frac{1}{13}\left(\frac{5\pi}{2} + 12\right)$

6. (16 points) Show the line integral below is independent of path and use the Fundamental Theorem for Line Integrals to evaluate.

$$\oint_{(-1,1)}^{(3,3)} \underbrace{(y^2 + x^2)}_{f}dx + \underbrace{(2xy + y^2)}_{g}dy$$

$\frac{\partial f}{\partial y} = 2y$ $\frac{\partial g}{\partial x} = 2y$ since = independent of path.

Let $\phi_x = y^2 + x^2 \Rightarrow \phi = xy^2 + \frac{x^3}{3} + h(y)$

$\phi_y = 2xy + y^2 \Rightarrow \phi = xy^2 + \frac{y^3}{3} + k(x)$

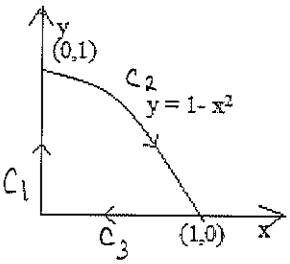
so $\phi = \frac{x^3}{3} + \frac{y^3}{3} + xy^2$

$\oint = \phi(3,3) - \phi(-1,1) = \frac{3^3}{3} + \frac{3^3}{3} + 3 \cdot 3^2 - \left(-\frac{1}{3} + \frac{1}{3} - 1\right) =$

$9 + 9 + 3 \cdot 9 + 1 = 46$

$\oint \vec{F}\cdot d\vec{r} = \int (x\hat{i} + x^2\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$

7. (17 points) Evaluate the line integral (which is also work done) $\oint_C \mathbf{F(r)}\cdot d\mathbf{r}$. Where $\mathbf{F(r)} = x\,\mathbf{i} + x^2\,\mathbf{j}$ and C is the closed path from (0,0) to (0,1) along y axis, along the curve $y = 1 - x^2$ from (0,1) to (1,0) and then along x axis from (1,0) to (0,0). Evaluate as a line integral, do not use Green's theorem.



path C_1: $x=0,\ y=t,\ 0\le t\le 1$. ∴ $\int \vec{F}\cdot d\vec{r} = 0$

path C_2: $x=t,\ y=1-t^2,\ 0\le t\le 1$. $dx=dt,\ dy=-2t\,dt$

$\oint F\cdot dr = \int_0^1 t\cdot dt + t^2(-2t\,dt) = \int_0^1 (t-2t^3)\,dt =$

$\left(\frac{t^2}{2} - \frac{t^4}{2}\right)\Big|_0^1 = 0$

path C_3: $x = (1-t),\ y=0,\ dx=-dt,\ dy=0$

$\int F\cdot dr = -\int_0^1 (1-t)\,dt + 0 = -\left(t - \frac{t^2}{2}\right)\Big|_0^1 = -\frac{1}{2}$

$\oint F\cdot dr = -\frac{1}{2}$

8. (17 points) Use Lagrange multipliers to find the maximum and minimum of $f(x,y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

$\nabla f = 5\hat{i} - 3\hat{j}$, $\nabla g = 2x\hat{i} + 2y\hat{j}$, $\nabla g = \lambda \nabla f$ (or $\nabla f = \lambda \nabla g$)

∴ $2x = 5\lambda$, $2y = -3\lambda$, so $\frac{2}{5}x = -\frac{2}{3}y$, $\frac{3}{5}x = -y$, $y = -\frac{3}{5}x$

So $x^2 + \left(-\frac{3}{5}x\right)^2 = 136$, $x^2 + \frac{9}{25}x^2 = 136$, $\frac{34}{25}x^2 = 136$

$x^2 = \frac{136\cdot 25}{34} = \frac{4\cdot 34\cdot 25}{34}$ or $x^2 = 10$; $x = \pm 10$, $y = \mp 6$

at $(10,-6)$, $f = 50 + 18 = 68$ Max

at $(-10,6)$, $f = -50 - 18 = -68$ min