Calculus III Test 4
Dr. Platte May 2, 2014 Name:______________________________

1. (17 points) Show the line integral below is independent of path and use the Fundamental Theorem for Line Integrals to evaluate.

$$\oint_{(1,1)}^{(2,3)} y\,dx + (x+2)\,dy$$

Find $\phi(x,y)$ s.t.

$\frac{\partial \phi}{\partial x} = y$, $\frac{\partial \phi}{\partial y} = x+2$

$f(x,y) = y$, $\frac{\partial f}{\partial y} = 1$

$g(x,y) = (x+2)$, $\frac{\partial g}{\partial x} = 1$

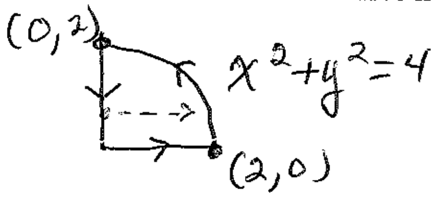
Since partials equal integral is indep. of path.

$\phi = xy + h_1(y)$, $\phi = xy + 2y + h_2(x)$

$\therefore \phi = xy + 2y$

$$\int_{(1,1)}^{(2,3)} y\,dx + (x+2)\,dy = (2\cdot 3 + 2\cdot 3) - (1\cdot 1 + 2\cdot 1) = 12 - 3 = 9$$

2. (17 points) Use Green's theorem to evaluate the following line integral. The curve (path) is the boundary of the region in the first quadrant enclosed by the x and y coordinate axes and the circle $x^2 + y^2 = 4$. Orientation of curve is counter clock wise.



Green's Theorem

$$\oint_C x^2\,dx + (x^2+y)\,dy = \iint_R \left(\frac{\partial(x^2+y)}{\partial x} - \frac{\partial x^2}{\partial y}\right) dA$$

$$= \iint_R 2x\,dA$$

$$\iint_R 2x\,dA = \int_0^2 \int_0^{\sqrt{4-y^2}} 2x\,dx\,dy$$

$$= \int_0^2 x^2 \Big|_0^{\sqrt{4-y^2}} dy = \int_0^2 (4-y^2)\,dy = \left(4y - \frac{y^3}{3}\right)\Big|_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

Note: Can convert to polar coordinates, but not necessary due to the x factor in the integrand.

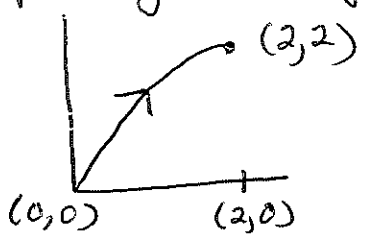
3. (17 points) Evaluate the line integral below, where path is $x = \frac{1}{2}y^2$ or $y^2 = 2x$ from (0, 0) to (2, 2).

$x = \frac{1}{2}y^2$, parabola opening to right

$$\int_C -y\,dx + x\,dy$$

$$= \int_0^2 -t\cdot t\,dt + \frac{1}{2}t^2\,dt = \int_0^2 -\frac{1}{2}t^2\,dt$$

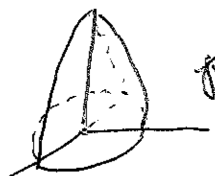
$$= -\frac{1}{2}\cdot\frac{1}{3}\cdot t^3\Big|_0^2 = -\frac{1}{2}\cdot\frac{1}{3}\cdot 8 = -\frac{4}{3}$$



For parameterization use $y = t$, $x = \frac{1}{2}t^2$, $0 \le t \le 2$

$dy = dt$, $dx = t\,dt$

4. (17 points) Use Stoke's Theorem to evaluate



paraboloid

$$\oint_C ((z+y)\boldsymbol{i} + (-x+z)\boldsymbol{j} + (x+y)\boldsymbol{k}) \cdot \boldsymbol{dr}$$

Boundary

$x^2+y^2=1$

Where the surface σ is the portion of the paraboloid $z = 1 - x^2 - y^2$ for which $z \ge 0$. So the curve is the boundary of the surface in the xy plane with a counter clock wise orientation.

$\sigma = 1 - x^2 - y^2 - z$, $\nabla\sigma$ is a normal to curve

$\nabla\sigma = -2x\,i - 2y\,j - k$

$\vec{F} = (z+y)i + (-x+z)j + (x+y)k$

$$\overrightarrow{\text{curl } F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z+y & -x+z & x+y \end{vmatrix}$$

$$\text{curl } F = (1-1)i + (1-1)j - (1+1)k$$
$$= -2k$$

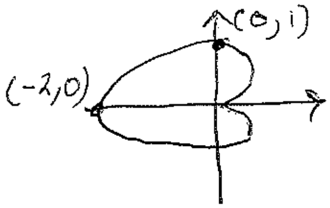
$$\text{so line integral} = \iint_R (2x\,i + 2y\,j + k)\cdot(-2k)\,dA$$

$$= \iint_R -2\,dA = -2\cdot(\pi\cdot 1^2) = -2\pi$$

Region is ○ of radius 1

5. (32 points) Given the cardioid $r = 1 - \cos(\theta)$.

a) Find the area of the region in the first quadrant bounded by the cardioid and the x and y axis.



area $= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} [1 - 2\cos\theta + \cos^2\theta] \, d\theta$

$= \frac{1}{2}\left[\theta - 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{3\pi}{4} - 2\right] = \frac{3\pi}{8} - 1$

b) Find the length of the arc of the cardioid in the first quadrant.
Hint: The trigonometric identity $1 - \cos(2\alpha) = 2\sin^2\alpha$ may be helpful.

arc Length $\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$

$\frac{dr}{d\theta} = \sin\theta$, so $r^2 + \left(\frac{dr}{d\theta}\right)^2 = [1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]$

$= 2(1-\cos\theta)$

Length $= \int_0^{\frac{\pi}{2}} \sqrt{2}\sqrt{1-\cos\theta} \, d\theta$, The identity yields $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

Note: $\sqrt{\sin^2\frac{\theta}{2}} = \left|\sin\frac{\theta}{2}\right|$, but $0 \le \theta \le \frac{\pi}{2}$ so sin is pos.

so Length $= \int_0^{\frac{\pi}{2}} \sqrt{2} \cdot \sqrt{2\sin^2\frac{\theta}{2}}$

$= 2\int_0^{\frac{\pi}{2}} \sin\frac{\theta}{2} \, d\theta = 2\left(-2\cos\frac{\theta}{2}\right)\Big|_0^{\frac{\pi}{2}}$

$= -4\left[\cos\frac{\pi}{4} - \cos 0\right] = -4\left[\frac{\sqrt{2}}{2} - 1\right]$

$= 4 - 2\sqrt{2}$