Calculus III　　Test IV
Dr. Platte　　April 30, 2018　　Name: Key

1. (20 points) Let f(x, y) = 8x, find value of the line integral $\int_C f\,ds$ along the parametrized curve C: x = t, y = t^2 from (0,0) to (1,1). (curve can also be given as r(x(t), y(t)) = t**i** + t^2**j** for $0 \le t \le 1$)

$\vec{r}(t) = t\,\hat{i} + t^2\hat{j}$, $r'(t) = \hat{i} + 2t\,\hat{j}$, $\|r'(t)\| = \sqrt{1+4t^2}$

$\int_0^1 f\,\|r'(t)\|\,dt = \int_0^1 8t\sqrt{1+4t^2}\,dt$, Let $u = 1+4t^2$, $du = 8t\,dt$

Limits $t=0, u=1$; $t=1, u=\sqrt{5}$

$\int_1^{\sqrt{5}} u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_1^{\sqrt{5}} = \frac{2}{3}\left[5\sqrt{5} - 1\right]$

2. (20 points) Show the line integral below is independent of path and use the Fundamental Theorem for Line Integrals to evaluate.

$$\oint_{(1,1)}^{(2,2)} (2xy+1)dx + (x^2+1)dy$$

For indep. of path $f_y = g_x$. Here $f = 2xy+1$, $f_y = 2x$
$g = x^2+1$, $g_x = 2x$. ∴ indep of path, thus ϕ exists.

$\phi_x = 2xy+1$　　　$\phi_y = x^2+1$
$\phi = x^2y + x + h_1(y)$　　　$\phi = yx^2 + y + h_2(x)$
So $\phi = x^2y + x + y$

Fund. Thm.
$\int_{(1,1)}^{(2,2)} (2xy+1)dx + (x^2+1)dy = (x^2y + x + y)\Big|_{(1,1)}^{(2,2)} =$

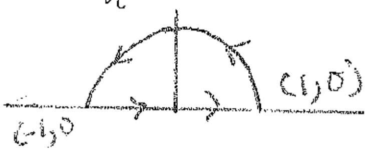
$(8+2+2) - (1+1+1) = 9$

3. (20 points) Use Green's theorem to evaluate the following line integral. The curve (path) is the boundary of the region is the upper half circle $x^2 + y^2 = 1$(in R^2) and x coordinate axes. Orientation of curve is counter clock wise.

$$\oint_C -y^3dx + (y + x^3)dy$$

$f = -y^3$, $f_y = -3y^2$

$g = y + x^3$, $g_x = 3x^2$

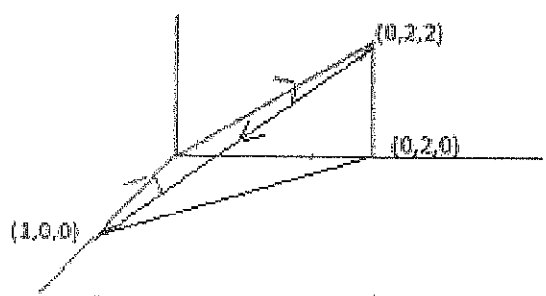


$= \iint_A (g_x - f_y)dA = \iint_A (3x^2 + 3y^2)dA$

Convert to polar

$$3\int_0^{\pi}\int_0^1 r^2\, r\,dr\,d\theta = 3\int_0^{\pi} \frac{r^4}{4}\Big|_0^1 d\theta = \frac{3}{4}\pi$$

4. (20 points) Using Stoke's theorem find the work performed by the force field $\mathbf{F}(x, y, z) = y\mathbf{i} + y^2z\mathbf{j} - x\mathbf{k}$ on a particle that traverses the triangle C in the plane $z = y$ shown in figure below.

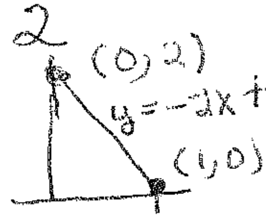


$$\overrightarrow{\text{curl}}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & y^2z & -x \end{vmatrix} = -y^2 i + 1j - k$$

surface $\sigma = z = y$

$\sigma_x = 0$, $\sigma_y = 1$

down $\vec{n} = \langle 0, 1, -1 \rangle$, $\overrightarrow{\text{curl}F} \cdot \vec{n} = 2$

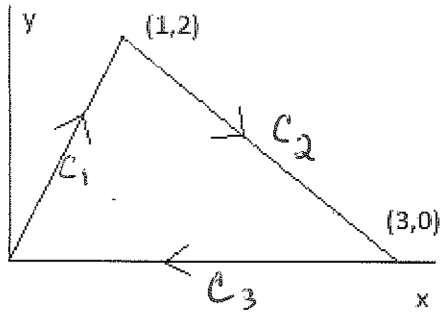


$A = \frac{1}{2} \cdot 1 \cdot 2 = 1$

$$\iint \overrightarrow{\text{curl}F} \cdot \vec{n}\,ds = \iint_\Delta 2\,dA$$

$$= 2\int_0^1 \int_0^{-2x+2} dy\,dx = 2 \cdot 1 = 2$$

5. (20 points) Evaluate the line integral below, where path is given in the figure. $\oint (y^2 dx + x dy)$



Path C_1: parallel vector $\hat{i}+2\hat{j}$

Line $x=t$, $y=2t$, $0 \le t \le 1$

$$\int_0^1 4t^2\,dt + t\,2dt = \tfrac{4}{3}t^3 + t^2\Big|_0^1 = \tfrac{7}{3}$$

Path C_2: // vector $2\hat{i}-2\hat{j}$, Line $x=1+2t$, $y=2-2t$, $0 \le t \le 1$

$$\int_0^1 (2-2t)^2\,2dt + (1+2t)(-2dt) = \int_0^1 \left(8-16t+8t^2-2-4t\right)dt$$

$$= \int_0^1 (6-20t+8t^2)\,dt = \left(6t-10t^2+\tfrac{8}{3}t^3\right)\Big|_0^1 = \tfrac{8}{3}-4 = -\tfrac{4}{3}$$

Path C_3: Line $x=3-3t$, $y=0$, $0 \le t \le 1$

So $dy=0$, $\int_0^1 = 0$.

So $\oint (y^2dx + xdy = \tfrac{7}{3} - \tfrac{4}{3} = 1$