Section 6.2: Confidence Intervals for the Mean (Small Sample s)

Today we will study:

- The t-distribution
- Construction of confidence intervals when \( n < 30 \)

The Student \( t \)-Distribution

Paraphrased from our text - section 6.2

The requirements for the methods of this section:

1. \( n < 30 \)
2. The sample is a simple random sample.
3. The sample is from a normally distributed population (or approximately normal)

We are still trying to estimate the population mean, as before the sample mean \( \bar{x} \) is the best point estimate of the population mean \( \mu \).

**DEFINITION**

If the distribution of a random variable \( x \) is approximately normal, then

\[
t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}
\]

follows a \( t \)-distribution. Critical values are denoted by \( t_c \). Several properties of the \( t \)-distribution are as follows.

1. The \( t \)-distribution is bell shaped and symmetric about the mean.
2. The \( t \)-distribution is a family of curves, each determined by a parameter called the degrees of freedom. The **degrees of freedom** are the number of free choices left after a sample statistic such as \( \bar{x} \) is calculated. When you use a \( t \)-distribution to estimate a population mean, the degrees of freedom are equal to one less than the sample size.
   
   \[ \text{d.f.} = n - 1 \]

   **Degrees of freedom**

3. The total area under a \( t \)-curve is 1 (or 100%) 
4. The mean, median, and mode of the \( t \)-distribution are equal to zero.
5. As the degrees of freedom increase, the \( t \)-distribution approaches the normal distribution. After 30 d.f. the \( t \)-distribution is very close to the standard normal \( z \)-distribution.

The tails in the \( t \)-distribution are “thicker” than those in the standard normal distribution.
The margin of error $E$ for the estimate of $\mu$
(For $n < 30$, $\sigma$ unknown, for an approximately normally distributed population)

$$E = t_c \frac{s}{\sqrt{n}}$$
where $t_c$ has $n - 1$ degrees of freedom

**Confidence Intervals and $t$-Distributions**

**GUIDELINES**

**Constructing a Confidence Interval for the Mean: $t$-distribution**

1. Identify the sample statistics $n$, $\bar{x}$, and $s$.
2. Identify the degrees of freedom, the level of confidence $c$, and the corresponding critical value $t_c$ (from the $t$-distribution table).
3. Find the margin of error $E$.

$$E = t_c \frac{s}{\sqrt{n}}$$
4. Find the left and right endpoints and write down the confidence interval.

Interval: $\bar{x} - E < \mu < \bar{x} + E$

The round-off rule is the same as in the previous section.

1. Round to the same number of decimal places given for the sample mean.
2. When using the original sample data - round to one more decimal place than used for the original data.
3. Do not round until the end (keep all the decimal places during intermediate steps).

**Interpreting a Confidence Interval**

Given: $a < \mu < b$

- **Correct**: We are 90% confident that the interval $(a, b)$ actually does contain the true value of $\mu$.
- **Wrong**: There is a 90% chance that the true value of $\mu$ will fall between $a$ and $b$.

**Remarks**

- The population mean $\mu$ exists and is a fixed value, either the limits enclose the value $\mu$ or they do not.
- See flowchart in the text for description of when to use the normal distribution to construct a confidence interval for the population mean and when to use a $t$-distribution.
- The level of confidence $c$ is the area under the standard normal curve between the critical values $-t_c$ and $t_c$.
- The area in the tails (outside $c$) is $1 - c$, so the area in each tail is $\frac{1}{2}(1 - c)$. 